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Poisson algebras for non-linear field theories in the Cahiers topos. **(English summary)**

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Classical field theory studies solutions to geometric partial differential equations (PDEs) on manifolds, which are usually endowed with some extra structures such as metrics and fiber bundles. If the PDE at issue arises as the Euler-Lagrange equation of some local Lagrangian, there is a canonical presymplectic form on the space of solutions [G. J. Zuckerman, in *Mathematical aspects of string theory (San Diego, Calif., 1986)*, 259–284, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987; MR0915825; I. Khavkine, Internat. J. Modern Phys. A **29** (2014), no. 5, 1430009; MR3189797]. An intriguing task is then to quantize the solution space along this presymplectic form, thereby achieving the transition from classical to quantum field theory.

The principal aim in this paper is to construct Poisson algebras for non-linear scalar field theories within a well-adapted model of synthetic differential geometry or, exactly speaking, the Cahiers topos for the sake of simplicity. The authors focus on a class of real scalar fields on Lorentzian manifolds with PDE given by the sum of the d'Alembert operator and a possibly non-polynomial interaction term, including Φ^4-theory as well as the sine-Gordon model, though the approach can be greatly generalized to more complicated non-linear field theories such as those formulated in terms of sections of generic vector bundles or even in terms of maps between smooth manifolds like the wave map equation (σ-model). The authors stick to scalar field theories in order to not obscure the construction of a Poisson algebra with more involved structures on the field theory side. It is to be stressed that the authors' construction of Poisson algebras for non-linear classical field theories does not depend on any PDE-analytical properties of the field equation or its linearization. The authors have succeeded in putting a clear splitting between abstract geometric/algebraic constructions of Poisson algebras (i.e., their existence) and PDE-analytical considerations afterwards, while such recent approaches as [R. Brunetti, K. Fredenhagen and P. L. Ribeiro, "Algebraic structure of classical field theory I: Kinematics and linearized dynamics for real scalar fields", preprint, arXiv:1209.2148] mix analytical, algebraic and geometric techniques from scratch.

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