STRONG GENERATORS OF THE SUBREGULAR $\mathcal{W}$-ALGEBRA $\mathcal{W}^{K-N}(\mathfrak{sl}_N, f_{sub})$ AND COMBINATORIAL DESCRIPTION AT CRITICAL LEVEL

NAOKI GENRA AND TOSHIRO KUWABARA

ABSTRACT. We construct explicitly strong generators of the affine $\mathcal{W}$-algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$ of subregular type A. Moreover, we are able to describe the OPEs between them at critical level. We also give a description the affine $\mathcal{W}$-algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ in terms of certain fermionic fields, which was conjectured by Adamović.

1. Introduction

For a reductive Lie algebra $\mathfrak{g}$, a nilpotent element $f \in \mathfrak{g}$ and $k \in \mathbb{C}$, the affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is defined as a vertex algebra constructed by the generalized quantum Drinfeld-Sokolov reduction; see [9, 18, 19, 20]. In this paper, we discuss the affine $\mathcal{W}$-algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$ associated with $\mathfrak{sl}_N$ and a subregular nilpotent element $f_{sub} \in \mathfrak{sl}_N$ with level $K_0 - N$, which we call the subregular $\mathcal{W}$-algebra. Recently, in [15, Section 6], the first author described the subregular $\mathcal{W}$-algebra by using certain screening operators, and showed that the subregular $\mathcal{W}$-algebra is isomorphic to a vertex algebra $W_N^{(2)}$ introduced by Feigin and Semikhatov in [10].

For a principal nilpotent element $f_{pr} \in \mathfrak{sl}_N$, the corresponding affine $\mathcal{W}$-algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{pr})$ is a vertex algebra such that, at critical level $K_0 = 0$, $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{pr})$ coincides with the center of affine vertex algebra $V^{-N}(\mathfrak{sl}_N)$, called the Feigin-Frenkel center. In [5, Section 2], Arakawa and Molev explicitly constructed strong generators of the vertex algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{pr})$. Their images through the Miura map are described by a certain noncommutative analog of the elementary symmetric polynomials, which recovers a result of Fateev and Lukyanov in [8]. They also constructed explicit strong generators of affine $\mathcal{W}$-algebras correponding to rectangular nilpotent elements in $\mathfrak{gl}_N$ ([5, Section 3]).

In Section 3 of this paper, we discuss construction of certain strong generators for the subregular $\mathcal{W}$-algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$. Our construction is based on the Feigin-Semikhatov description of $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$, which describes $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$ as intersection of the kernels of screening operators on a certain lattice vertex algebra. We construct elements W_m $(m = 2, \ldots, N)$ of the lattice vertex algebra by using the noncommutative elementary symmetric polynomials of the elements of the Heisenberg part of the lattice vertex algebra, and show that they lie in the intersection of the kernels of the screening operators (Definition 3.8 and Proposition 3.5). We also show that these elements are algebraically independent in the Zhu's C_2 Poisson algebra of the vertex algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$, and it implies that the elements $W_2, \ldots, W_{N-1}$, together with the generators E, H, F of Feigin-Semikhatov's, strongly generate $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$ (Theorem 3.14).

In Section 4, we discuss the subregular $\mathcal{W}$-algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ at critical level, $K_0 = 0$. At critical level, the vertex algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ has a nontrivial

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center as a vertex algebra, and the center is naturally isomorphic to the Feigin-Frenkel center of the affine vertex algebra $V^{-N}(\mathfrak{sl}_N)$. We show that our elements $W_2, \dots, W_N$ strongly generate the center (Proposition 4.2). Moreover, we give an explicit form of OPEs between the strong generators (Theorem 4.4).

In [1], Adamović conjectured that the subregular $\mathcal{W}$-algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ is isomorphic to a vertex algebra generated by certain fields, consisting of certain fermionic fields and the generators of the Feigin-Frenkel center. We prove his conjecture by using our strong generators (Theorem 4.5).

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2. Subregular $\mathcal{W}$-algebras and Feigin-Semikhatov screenings

The subregular $\mathcal{W}$-algebra $\mathcal{W}^{K_0} = \mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$ at level $K_0 - N$ is a vertex algebra defined by the generalized quantum Drinfeld-Sokolov reduction associated with $\mathfrak{sl}_N, f_{sub}$ and $K_0 \in \mathbb{C}$ [18], where $f_{sub} = e_{-\alpha_2} + \dots + e_{-\alpha_{N-1}} \in \mathfrak{sl}_N$ is a subregular nilpotent element in $\mathfrak{sl}_N$. We introduce a free field realization of $\mathcal{W}^{K_0}$ following [10] and [15].

We follow [12, 17] for definitions of vertex algebras, and denote by $A(z) = Y(A, z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$ a field on V for an element A in a vertex algebra V. Let K be an indeterminate and $\mathcal{W}^K_{\mathbb{C}[K]}$ the subregular $\mathcal{W}$-algebra associated with $\mathfrak{sl}_N, f_{sub}, K$ over $\mathbb{C}[K]$. From the proof of a vanishing theorem of the BRST cohomology in [19, 20] (for $\mathcal{W}^{K_0}$, but the same proof can be applied for $\mathcal{W}^K_{\mathbb{C}[K]}$), it follows that $\mathcal{W}^K_{\mathbb{C}[K]} \otimes \mathbb{C}_{K_0} = \mathcal{W}^{K_0}$. Here, $\mathbb{C}_{K_0} = \mathbb{C}[K]/(K - K_0) \simeq \mathbb{C}$ is a one-dimensional $\mathbb{C}[K]$-module on which K acts by K_0. See e.g. [4]. Let $\mathbb{V} = \mathbb{C}[K]A_{N-1} \oplus \dots \oplus \mathbb{C}[K]A_1 \oplus \mathbb{C}[K]Q \oplus \mathbb{C}[K]Y$ be a free $\mathbb{C}[K]$-module of rank $N+1$. We define a symmetric bilinear form on $\mathbb{V}$ given by the Gram matrix

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} A_{N-1} & A_{N-2} & A_{N-3} & \cdots & \cdots & A_2 & A_1 & Q & Y \end{array} \\
\begin{array}{c} A_{N-1} \\ A_{N-2} \\ A_{N-3} \\ \vdots \\ \vdots \\ A_2 \\ A_1 \\ Q \\ Y \end{array} &
\begin{pmatrix}
2K & -K & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-K & 2K & -K & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & -K & 2K & -K & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 2K & -K & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & -K & 2K & -K & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -K & 1 & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}.
$$

Consider a Heisenberg vertex algebra $\mathcal{H}^K$ over $\mathbb{C}[K]$ associated with the bilinear form on $\mathbb{V}$. It is a vertex algebra generated by the elements A_i, Q and Y ($i = 1, \dots, N-1$) subject to the OPE $a(z)b(w) \sim (a, b)/(z-w)^2$ where $a, b = A_i$, Q or Y, and $(\ ,\ )$ is the bilinear form on $\mathbb{V}$. For a vector $w \in \mathbb{V}$, let $\mathcal{H}^K_w$ be the Fock representation of $\mathcal{H}^K$ of heighest weight $(w, —)$. See [12, Section 5.4.1]. We denote the highest weight vector of $\mathcal{H}^K_w$ by $e^{\int w}$. The direct sum $\mathcal{V}^K_{\mathbb{C}[K]} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^K_{mY}$ is equipped with a vertex algebra structure. Indeed, the vertex operator $e^{m\int Y}(z) = \sum_n e^{m\int Y}_{(n)} z^{-n-1}$ corresponding to the highest weight vector $e^{m\int Y}$ is a field with the following OPEs

$$e^{m\int Y}(z)e^{n\int Y}(w) \sim 0, \quad a(z)e^{m\int Y}(w) \sim \frac{(a, mY)}{z-w} e^{m\int Y}(w)$$

for m, $n \in \mathbb{Z}$, and $a \in \mathbb{V}$ and the derivative of $e^{m\int Y}(z)$ is given by $\partial e^{m\int Y}(z) = {}^\circ_\circ mY(z)e^{m\int Y}(z){}^\circ_\circ$.

Similarly to $e^{\int Y}(z)$, we also have the vertex operator $e^{\int Q}(z)$ (resp. $e^{\int A_i}(z)$) associated with $Q \in \mathbb{V}$ (resp. $A_i \in \mathbb{V}$ for $i = 1, \ldots, N-1$). The OPEs between these vertex operators and fields of $\mathcal{V}^K_{\mathbb{C}[K]}$ are given as follows:

$$\begin{aligned} &a(z)e^{\int b}(w) \sim \frac{(a, b)}{z-w} e^{\int b}(w), \quad e^{\int A_i}(z)e^{m\int Y}(w) \sim 0, \\ &e^{\int Q}(z)e^{m\int Y}(w) \sim (z-w)^m e^{m\int Y + \int Q}(w), \end{aligned}$$

where a, $b \in \mathbb{V}$, $i = 1, \ldots, N-1$ and $m \in \mathbb{Z}$. The residue of $e^{\int Q}(z)$ (resp. $e^{\int A_i}(z)$) gives an operator on $\mathcal{V}^K_{\mathbb{C}[K]}$ such that $e^{\int Q}_{(0)} : \mathcal{H}^K_{mY} \longrightarrow \mathcal{H}^K_{mY+Q}$ (resp. $e^{\int A_i}_{(0)} : \mathcal{H}^K_{mY} \longrightarrow \mathcal{H}^K_{mY+A_i}$) for $m \in \mathbb{Z}$ and $i = 1, \ldots, N-1$. The operators $e^{\int Q}_{(0)}$, $e^{\int A_i}_{(0)}$ are called screening operators. The vertex algebra given as intersection of the kernels of these screening operators were introduced by Feigin and Semikhatov in [10]. Recently, the first author showed that their vertex algebra is isomorphic to the subregular $\mathcal{W}$-algebra.

Proposition 2.1 ([15], Theorem 6.9)**.** *As a vertex algebra over the ring $\mathbb{C}[K]$, we have an isomorphism*

$$\mu^K : \mathcal{W}^K_{\mathbb{C}[K]} \xrightarrow{\sim} \mathrm{Ker}\, e^{\int Q}_{(0)} \cap \bigcap_{i=1}^{N-1} \mathrm{Ker}\, e^{\int A_i}_{(0)}.$$

Since $\mathcal{W}^K_{\mathbb{C}[K]} \otimes \mathbb{C}_{K_0} = \mathcal{W}^{K_0}$, we have an embedding $\mathcal{W}^{K_0} \hookrightarrow \mathcal{V}^{K_0}$ for all $K_0 \in \mathbb{C}$. We remark that the embedding is obtained as the composition of three maps $\mu, \mu_W, \mu_{\beta\gamma}$ defined as follows. Applying the specialization functor $? \otimes \mathbb{C}_{K_0}$ to embeddings

$$\mathrm{Ker}\, e^{\int Q}_{(0)} \cap \bigcap_{i=1}^{N-1} \mathrm{Ker}\, e^{\int A_i}_{(0)} \to \mathrm{Ker}\, e^{\int Q}_{(0)} \cap \mathrm{Ker}\, e^{\int A_1}_{(0)} \to \mathrm{Ker}\, e^{\int Q}_{(0)} \to \mathcal{V}^K_{\mathbb{C}[K]},$$

we have vertex algebra homomorphisms

$$\begin{aligned} \mathcal{W}^{K_0} \xrightarrow{\mu} V^{\tau_{K_0-N}}(\mathfrak{g}_0) \simeq V^{K_0-2}(\mathfrak{sl}_2) \otimes V^{K_0}(\mathbb{C}^{N-2}) \\ \xrightarrow{\mu_W} \mathcal{D}^{ch}(\mathbb{C}^1) \otimes V^{K_0}(\mathbb{C}^{N-1}) \xrightarrow{\mu_{\beta\gamma}} \mathcal{V}^{K_0}, \end{aligned}$$

where $V^{\tau_{K_0-N}}(\mathfrak{g}_0)$ is the affine vertex algebra associated with the Lie subalgebra $\mathfrak{g}_0$ and the bilinear form τ_{K_0-N} on $\mathfrak{g}_0$ defined in [15, (2.2)], and $\mathcal{D}^{ch}(\mathbb{C}^1)$ is the vertex algebra of $\beta\gamma$-system of rank one. It then follows that $\mu, \mu_W, \mu_{\beta\gamma}$ are injective maps, called the Miura map for $\mathcal{W}^{K_0}$ [19, 15], Wakimoto realization for $V^{K_0-2}(\mathfrak{sl}_2)$ [24, 11] and Friedan-Martinec-Shenker bosonization [14] respectively.

3. Strong generators of the subregular $\mathcal{W}$-algebra

In this section, we explicitly construct elements $W_2, \ldots, W_N$ of the subregular $\mathcal{W}$-algebra, and show that $E, F, H, W_2, \ldots, W_{N-1}$ strongly generate the vertex algebra $\mathcal{W}^{K_0} = \mathcal{W}^K_{\mathbb{C}[K]} \otimes \mathbb{C}_{K_0}$ for any $K_0 \in \mathbb{C}$. We also show that $W_2, \ldots, W_N$ generate the Poisson center in Zhu's C_2-Poisson algebra of $\mathcal{W}^{K_0}$ and they are algebraically independent.

Set $\ell_N(K) = K(N-1)/N - 1$. Define three elements in $\mathcal{V}^K_{\mathbb{C}[K]}$

$$H = \ell_N(K)Y + Q + \frac{N-1}{N}A_1 + \cdots + \frac{2}{N}A_{N-2} + \frac{1}{N}A_{N-1},$$
$$E = e^{\int Y}, \qquad F = -\rho_N \cdots \rho_2\rho_1 e^{-\int Y},$$

where $\rho_i = (K-1)(\partial + Y_{(-1)}) + Q_{(-1)} + \sum_{j=1}^{i-1} A_{j(-1)}$ for $i = 1, \ldots, N$. Note that $\rho_1 e^{-\int Y} = Q_{(-1)}e^{-\int Y}$ because $\partial e^{-\int Y} = -Y_{(-1)}e^{-\int Y}$. In [10], Feigin and Semikhatov showed that these three elements generate the vertex algebra $\mathcal{W}^K_{\mathbb{C}[K]}$. The first goal of the present paper is to construct a set of strong generators of $\mathcal{W}^K_{\mathbb{C}[K]}$, including these three elements H, E and F.

Define N elements in the Heisenberg part $\mathcal{H}^K$ of the vertex algebra $\mathcal{V}^K_{\mathbb{C}[K]}$

$$X_i = -\frac{K}{N}Y - \sum_{j=1}^{i-1}\frac{j}{N}A_j + \sum_{j=i}^{N-1}\frac{N-j}{N}A_j \in \mathcal{H}^K$$

for $i = 1, \ldots, N$. Then, we have $\rho_i = (K-1)\partial + H_{(-1)} - X_{i(-1)}$ for $i = 1, \ldots, N$. Also, we define

$$X_0 = -\frac{K}{N}Y + Q + \frac{N-1}{N}A_1 + \cdots + \frac{1}{N}A_{N-1} \in \mathcal{H}^K \subset \mathcal{V}^K_R$$

and $\rho_0 = (K-1)\partial + H_{(-1)} - X_{0(-1)} = (K-1)(\partial + Y_{(-1)})$.

Recall the definition of the noncommutative elementary symmetric polynomials (cf. [21, (12.48)]). Let $\xi_1, \ldots, \xi_N$ be mutually noncommutative N operators on a certain vector space. Define the m-th noncommutative elementary symmetric polynomial in $\xi_1, \ldots, \xi_N$,

$$e_m(\xi_1, \ldots, \xi_n) = \sum_{i_1 > i_2 > \cdots > i_m} \xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}.$$

Note that we arrange the operators reverse-lexicographically.

We extend the vertex algebra $\mathcal{W}^K_{\mathbb{C}[K]}$ (resp. $\mathcal{V}^K_{\mathbb{C}[K]}$) by the commutative ring $R := \mathbb{C}[K, (K-1)^{-1}]$, and write $\mathcal{W}^K_R = \mathcal{W}^K_{\mathbb{C}[K]} \otimes_{\mathbb{C}[K]} R$ (resp. $\mathcal{V}^K_R = \mathcal{V}^K_{\mathbb{C}[K]} \otimes_{\mathbb{C}[K]} R$). First, we define elements of $\mathcal{V}^K_R$ and prove that they lie in $\mathcal{W}^K_R$. Later, we will normalize them to construct elements of $\mathcal{W}^K_{\mathbb{C}[K]}$. For $m = 1, \ldots, N$, we define an element of $\mathcal{V}^K_R$

$$W'_m = \sum_{k=0}^{N}(-1)^k\Big(\prod_{j=1}^{k}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{1\le i_1<\cdots<i_{N-k}\le N} e_m(\overbrace{\rho_0, \ldots, \rho_0}^{k\text{-times}}, \rho_{i_1}, \ldots, \rho_{i_{N-k}}). \tag{1}$$

We also introduce the generating function of these elements W'_m ($m = 1, \ldots, N$). Let u be an indeterminate which commutes with all other elements. Note that, for operators $\xi_1, \ldots, \xi_N$, we have

$$(u + \xi_N)\cdots(u + \xi_1) = \sum_{m=0}^{N} e_m(\xi_1, \ldots, \xi_N)\, u^{N-m}$$

by the definition of e_m. Setting $\rho_i(u) = u + \rho_i$, we have

$$\begin{aligned}(2)\quad \widetilde{W}^{(N)}(u) &:= \sum_{m=0}^{N} W'_m u^{N-m}\\ &= \sum_{k=0}^{N}(-1)^k\Big(\prod_{j=1}^{k}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{N\geq i_1>\cdots>i_{N-k}\geq 1}\rho_{i_1}(u)\cdots\rho_{i_{N-k}}(u)\rho_0(u)^k\mathbf{1},\end{aligned}$$

where W'_0 is equal to $\mathbf{1}$ up to multiplication by a certain constant.

We will show that the elements W'_m $(m = 1, \dots, N)$ belong to the subregular $\mathcal{W}$-algebra $\mathcal{W}^K_R$ by calculating the action of the screening operators $e^{\int Q}_{(0)}$, $e^{\int A_i}_{(0)}$ $(i = 1, \dots, N-1)$.

Lemma 3.1. *(1) For $i = 1, \dots, N-1$, $j = 0, \dots, N$ and $m \in \mathbb{Z}$, we have*

$$[\, e^{\int A_i}_{(m)}, \rho_j(u)\,] = \begin{cases}(m(K-1)+K)e^{\int A_i}_{(m-1)} & (j = i),\\ (m(K-1)-K)e^{\int A_i}_{(m-1)} & (j = i+1),\\ m(K-1)e^{\int A_i}_{(m-1)} & (j \neq i, i+1)\end{cases}$$

in $\bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}[K]}(\mathcal{H}^K_{mY}, \mathcal{H}^K_{mY+A_i})[u]$.

(2) For $j = 0, \dots, N$ and $m \in \mathbb{Z}$, we have

$$[\, e^{\int Q}_{(m)}, \rho_j(u)\,] = \begin{cases}(m-1)(K-1)e^{\int Q}_{(m-1)} & (j = 0),\\ (m(K-1)-K)e^{\int Q}_{(m-1)} & (j = 1),\\ m(K-1)e^{\int Q}_{(m-1)} & (j = 2, \dots, N)\end{cases}$$

in $\bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}[K]}(\mathcal{H}^K_{mY}, \mathcal{H}^K_{mY+Q})[u]$.

Proof. Note that $(A_i, X_i) = K$, $(A_i, X_{i+1}) = -K$, $(A_i, X_j) = 0$ for $j \neq i, i+1$, and $(Q, X_0) = -K+1$, $(Q, X_1) = -K$, $(Q, X_j) = 0$ for $j = 2, \dots, N$. Also, we have $(A_i, H) = (Q, H) = 0$ for any $i = 1, \dots, N-1$. Then, both (1) and (2) can be checked by direct computation. □

Proposition 3.2. *For $i = 1, \dots, N-1$, we have $e^{\int A_i}_{(0)}\widetilde{W}^{(N)}(u) = 0$.*

Proof. First, note that we have

$$e^{\int A_i}_{(0)}\rho_N(u)\cdots\rho_{i+2}(u)\rho_{i+1}(u)\cdots\mathbf{1} = \rho_N(u)\cdots\rho_{i+2}(u)e^{\int A_i}_{(0)}\rho_{i+1}(u)\cdots\mathbf{1}$$

since the screening operator $e^{\int A_i}_{(0)}$ commutes with $\rho_j(u)$ for $j \neq i$, $i+1$ by Lemma 3.1.

In $\widetilde{W}^{(N)}(u)$, there exists three kinds of terms; i) $\cdots\rho_{i+1}(u)\rho_i(u)\cdots\mathbf{1}$, terms with both factors $\rho_{i+1}(u)$ and $\rho_i(u)$, ii) $\cdots\rho_{i+1}(u)\cdots\mathbf{1}$ or $\cdots\rho_i(u)\cdots\mathbf{1}$, terms with either $\rho_{i+1}(u)$ or $\rho_i(u)$, iii) $\cdots(\rho_{i+1}(u))^\wedge(\rho_i(u))^\wedge\cdots\mathbf{1}$, terms without $\rho_{i+1}(u)$ nor $\rho_i(u)$. We consider the action of $e^{\int A_i}_{(0)}$ in these three cases individually.

i) By Lemma 3.1 (1), we have

$$\begin{aligned}&e^{\int A_i}_{(0)}\cdots\rho_{i+1}(u)\rho_i(u)\cdots\mathbf{1}\\ &= \cdots[\, e^{\int A_i}_{(0)}, \rho_{i+1}(u)\,]\rho_i(u)\cdots\mathbf{1} + \cdots\rho_{i+1}(u)[\, e^{\int A_i}_{(0)}, \rho_i(u)\,]\cdots\mathbf{1}\\ &= K\cdots(\rho_{i+1}(u)-\rho_i(u))e^{\int A_i}_{(-1)}\cdots\mathbf{1} \;-\; K\cdots((-1)(K-1)+K)e^{\int A_i}_{(-2)}\cdots\mathbf{1}\\ &= K\cdots\big\{A_{i(-1)}e^{\int A_i}_{(-1)} - e^{\int A_i}_{(-2)}\big\}\cdots\mathbf{1} = 0.\end{aligned}$$

Here we used $[e^{\int A_i}_{(m)}, H_{(-1)} - X_{j(n)}] = 0$ for $j \neq i,\ i+1$ with $m,\ n \in \mathbb{Z}$, and $e^{\int A_i}_{(-2)} = \left(A_{i(-1)} e^{\int A_i}\right)_{(-1)}$.

ii) Note that the generating function $\widetilde{W}^{(N)}(u)$ is symmetric by permutation between factors $\rho_1(u), \ldots, \rho_N(u)$. This implies that, for a term $(\cdots)_1 \rho_{i+1}(u)(\cdots)_2 \mathbf{1}$ of type ii), we also have a term $(\cdots)_1 \rho_i(u)(\cdots)_2 \mathbf{1}$ of exactly the same form except for replacing $\rho_{i+1}(u)$ by $\rho_i(u)$. Here the factors different from $\rho_{i+1}(u)$ and $\rho_i(u)$ are denoted by $(\cdots)_j$ $(j = 1, 2)$. Then, by Lemma 3.1 (1), we have

$$
\begin{aligned}
& e^{\int A_i}_{(0)} \left\{ (\cdots)_1 \rho_{i+1}(u)(\cdots)_2 \mathbf{1} + (\cdots)_1 \rho_i(u)(\cdots)_2 \mathbf{1} \right\} \\
&= (\cdots)_1 \left(-K e^{\int A_i}_{(-1)}\right)(\cdots)_2 \mathbf{1} + (\cdots)_1 \left(+K e^{\int A_i}_{(-1)}\right)(\cdots)_2 \mathbf{1} = 0.
\end{aligned}
$$

iii) A term without $\rho_{i+1}(u)$ and $\rho_i(u)$ trivially vanishes by the action of the screening operator $e^{\int A_i}_{(0)}$ by Lemma 3.1 (1).

As a consequence, we have $e^{\int A_i}_{(0)} \widetilde{W}^{(N)}(u) = 0$. □

The action of another screening operator $e^{\int Q}_{(0)}$ is more complicated. Previous to the calculation of $e^{\int Q}_{(0)} \widetilde{W}^{(N)}(u)$ we prepare the following lemma.

Lemma 3.3. *For $m \geq 0$, we have*

$$
e^{\int Q}_{(0)} \left(\rho_1(u)\rho_0(u)^m \mathbf{1} - \frac{(m+1)(K-1)+1}{(m+1)(K-1)} \rho_0(u)^{m+1} \mathbf{1} \right) = 0.
$$

Proof. To prove the equality of the lemma, note that

$$
\begin{aligned}
[\, e^{\int Q}_{(0)}, \rho_0(u)^l \,] &= \sum_{i=1}^{l} (-1)^i (K-1)^i \frac{l!}{(l-i)!} \rho_0(u)^{l-i} e^{\int Q}_{(-i)}, \\
[\, e^{\int Q}_{(-1)}, \rho_0(u)^l \,] &= \sum_{i=2}^{l+1} (-1)^{i-1} (K-1)^{i-1} \frac{l!}{(l-i+1)!} i \rho_0(u)^{l-i+1} e^{\int Q}_{(-i)},
\end{aligned}
$$

by Lemma 3.1 (2).

First we deal with the first term of the equality of the lemma. Using the fact $\rho_1(u) = \rho_0(u) + Q_{(-1)}$ and the identity

$$
Q_{(-1)} \rho_0(u)^{m-i} = \sum_{j=0}^{m-i} (-1)^j \frac{(m-i)!}{(m-i-j)!} \rho_0(u)^{m-i-j} Q_{(-j-1)},
$$

we obtain

$$
\begin{aligned}
(3)\quad & e^{\int Q}_{(0)}\rho_1(u)\rho_0(u)^m\mathbf{1} = \rho_1(u)e^{\int Q}_{(0)}\rho_0(u)^m\mathbf{1} - Ke^{\int Q}_{(-1)}\rho_0(u)^m\mathbf{1}\\
&= -K\rho_0(u)^m e^{\int Q}_{(-1)}\mathbf{1}\\
&\qquad + K\sum_{i=2}^{m+1}(-1)^i(K-1)^{i-1}\frac{m!}{(m-i+1)!}i\rho_0(u)^{m-i+1}e^{\int Q}_{(-i)}\mathbf{1}\\
&\qquad + \sum_{i=1}^{m}(-1)^i(K-1)^i\frac{m!}{(m-i)!}\rho_1(u)\rho_0(u)^{m-i}e^{\int Q}_{(-i)}\mathbf{1}\\
&= \sum_{i=1}^{m+1}(-1)^i(K-1)^{i-1}\frac{m!}{(m-i+1)!}i\rho_0(u)^{m-i+1}e^{\int Q}_{(-i)}\mathbf{1}\\
&\qquad + \sum_{i=1}^{m}(-1)^i(K-1)^i\frac{(m+1)!}{(m-i+1)!}\rho_0(u)^{m-i+1}e^{\int Q}_{(-i)}\mathbf{1}\\
&\qquad + \sum_{i=1}^{m}(-1)^i(K-1)^i\frac{m!}{(m-i)!}Q_{(-1)}\rho_0(u)^{m-i}e^{\int Q}_{(-i)}\mathbf{1}\\
&= \sum_{i=1}^{m+1}(-1)^i(K-1)^{i-1}\frac{m!}{(m-i+1)!}i\rho_0(u)^{m-i+1}e^{\int Q}_{(-i)}\mathbf{1}\\
&\qquad + \sum_{i=1}^{m}(-1)^i(K-1)^i\frac{(m+1)!}{(m-i+1)!}\rho_0(u)^{m-i+1}e^{\int Q}_{(-i)}\mathbf{1}\\
&\qquad + \sum_{i=1}^{m}\sum_{j=0}^{m-i}(-1)^{i+j}(K-1)^{i+j}\frac{m!}{(m-i-j)!}\rho_0(u)^{m-i-j}Q_{(-j-1)}e^{\int Q}_{(-i)}\mathbf{1}.
\end{aligned}
$$

Next, for the second term of the equality of the lemma, we have

$$
\begin{aligned}
(4)\quad & e^{\int Q}_{(0)}\frac{(m+1)(K-1)+1}{(m+1)(K-1)}\rho_0(u)^{m+1}\mathbf{1}\\
&= \sum_{i=1}^{m+1}(-1)^i(K-1)^i\frac{(m+1)!}{(m-i+1)!}\rho_0(u)^{m+1-i}e^{\int Q}_{(-i)}\mathbf{1}\\
&\quad + \sum_{i=1}^{m+1}(-1)^i(K-1)^{i-1}\frac{m!}{(m-i+1)!}\rho_0(u)^{m+1-i}e^{\int Q}_{(-i)}\mathbf{1}.
\end{aligned}
$$

The second term in the RHS of (3) and the first term in the RHS of (4) are canceled out. Since $\partial e^{\int Q} = Q_{(-1)}e^{\int Q}$, we have

$$\sum_{j=0}^{k} Q_{(-j-1)}e^{\int Q}_{(-k+j)}\mathbf{1} = ke^{\int Q}_{(-k-1)}\mathbf{1}$$

for $k \geq 1$. Thus

$$
\begin{aligned}
&e^{\int Q}_{(0)}\Big(\rho_1(u)\rho_0(u)^m\mathbf{1} - \frac{(m+1)(K-1)+1}{(m+1)(K-1)}\rho_0(u)^{m+1}\mathbf{1}\Big)\\
&\qquad = \sum_{i=2}^{m+1}(-1)^i(K-1)^{i-1}\frac{m!}{(m-i+1)!}(i-1)\rho_0(u)^{m-i+1}e^{\int Q}_{(-i)}\mathbf{1}\\
&\qquad + \sum_{k=1}^{m}\sum_{j=0}^{k}(-1)^k(K-1)^k\frac{m!}{(m-k)!}\rho_0(u)^{m-k}Q_{(-j-1)}e^{\int Q}_{(-k+j)}\mathbf{1}\\
&\qquad = \sum_{i=2}^{m+1}(-1)^i(K-1)^{i-1}\frac{m!}{(m-i+1)!}(i-1)\rho_0(u)^{m-i+1}e^{\int Q}_{(-i)}\mathbf{1}\\
&\qquad\qquad + \sum_{k=1}^{m}(-1)^k(K-1)^k\frac{m!}{(m-k)!}\rho_0(u)^{m-k}ke^{\int Q}_{(-k-1)}\mathbf{1} = 0.
\end{aligned}
$$

□

Proposition 3.4. *We have* $e^{\int Q}_{(0)}\widetilde{W}^{(N)}(u) = 0$.

Proof. We split terms in the definition of $\widetilde{W}^{(N)}(u)$ into two parts; terms with the factor $\rho_1(u)$ and terms without $\rho_1(u)$. Using the fact that the screening operator $e^{\int Q}_{(0)}$ commutes with $\rho_i(u)$ for all $i \neq 0,\,1$, together with Lemma 3.3, we calculate

$$
\begin{aligned}
&e^{\int Q}_{(0)}\widetilde{W}^{(N)}(u)\\
&= e^{\int Q}_{(0)}\sum_{m=0}^{N}(-1)^m\Big(\prod_{j=1}^{m}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{N\geq i_1>\cdots>i_{N-m}\geq 1}\rho_{i_1}(u)\cdots\rho_{i_{N-m}}(u)\rho_0(u)^m\mathbf{1}\\
&= e^{\int Q}_{(0)}\Big\{\sum_{m=0}^{N}(-1)^m\Big(\prod_{j=1}^{m}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{N\geq i_1>\cdots>i_{N-m-1}\geq 2}\rho_{i_1}(u)\cdots\rho_{i_{N-m-1}}(u)\rho_1(u)\rho_0(u)^m\mathbf{1}\\
&+ \sum_{m=0}^{N}(-1)^{m+1}\Big(\prod_{j=1}^{m+1}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{N\geq i_1>\cdots>i_{N-m-1}\geq 2}\rho_{i_1}(u)\cdots\rho_{i_{N-m-1}}(u)\rho_0(u)^{m+1}\mathbf{1}\Big\} = 0
\end{aligned}
$$

as desired. □

By Proposition 3.2 and Proposition 3.4, we have $W'_m \in \operatorname{Ker} e^{\int Q}_{(0)} \cap \bigcap_{i=1}^{N-1}\operatorname{Ker} e^{\int A_i}_{(0)}$ for $m = 1,\,\dots,\,N$. Now, we have the following proposition as a consequence of Proposition 2.1.

Proposition 3.5. *For $m = 1,\,\dots,\,N$, we have $W'_m \in \mathcal{W}^K_R$.*

To construct elements of the vertex algebra $\mathcal{W}^K_{\mathbb{C}[K]}$, we need to normalize the elements W'_m for $m = 2,\,\dots,\,N$.

Lemma 3.6. *Let q be an indeterminate. For $l,\,m \geq k$, we have the following identity:*

$$
\sum_{l=k}^{N}(-1)^{l-k}\binom{N-m+k}{l}\binom{l}{k}\prod_{j=k+1}^{l}\Big(1+\frac{q}{j}\Big) = \frac{(-1)^{N-m}}{(N-m)!}\prod_{j=0}^{N-m-1}(q-j).
$$

Proof. By direct calculation, we have

$$\begin{aligned}
&\sum_{l=k}^{N}(-1)^{l-k}\binom{N-m+k}{l}\binom{l}{k}\prod_{j=k+1}^{l}\Big(1+\frac{q}{j}\Big)\\
&=\sum_{l=0}^{N-m}(-1)^{l}\binom{N-m+k}{l+k}\frac{1}{l!}\prod_{j=1}^{l}(q+k+j)\\
&=\frac{(N-m+k)!}{(N-m)!k!}\sum_{l=0}^{N-m}\frac{(-N+m)_l(q+k+1)_l}{(k+1)_l}\frac{1^l}{l!},
\end{aligned}$$

where $(x)_l=\prod_{j=0}^{l-1}(x+j)$. The RHS can be described in terms of the hypergeometric function ${}_2F_1(a,b,c;z)$, and then in terms of the Γ-function $\Gamma(z)$ using Gauss's hypergeometric theorem. Thus

$$\begin{aligned}
&\sum_{l=k}^{N}(-1)^{l-k}\binom{N-m+k}{l}\binom{l}{k}\prod_{j=k+1}^{l}\Big(1+\frac{q}{j}\Big)\\
&=\frac{(N-m+k)!}{(N-m)!k!}\,{}_2F_1(-N+m,q+k+1,k+1;1)\\
&=\frac{(N-m+k)!}{(N-m)!k!}\,\frac{\Gamma(k+1)\Gamma(N-m-q)}{\Gamma(N-m+k+1)\Gamma(-q)}\\
&=\frac{(N-m+k)!}{(N-m)!k!}\,\frac{k!}{(N-m+k)!}\prod_{j=0}^{N-m-1}(-q+j)=\frac{(-1)^{N-m}}{(N-m)!}\prod_{j=0}^{N-m-1}(q-j).
\end{aligned}$$

□

Lemma 3.7. *Let ξ_0, ξ_1, ..., ξ_N be operators. For $m=1$, ..., N, we have*

$$\begin{aligned}
&\sum_{k=0}^{N}(-1)^k\Big(\prod_{j=1}^{k}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{1\le i_1<\cdots<i_{N-k}\le N}e_m(\overbrace{\xi_0,\dots,\xi_0}^{k\text{-times}},\xi_{i_1},\dots,\xi_{i_{N-k}})\\
&\quad=\Big(\prod_{j=1}^{N-m}\frac{j(K-1)-K}{j(K-1)}\Big)\sum_{k=0}^{m}(-1)^k\Big(\prod_{j=1}^{k}\frac{j(K-1)+1}{j(K-1)}\Big)e_{m-k}(\xi_1,\dots,\xi_N)\xi_0^k.
\end{aligned}$$

Proof. First, we have (below, we write $e_{m-k}(\xi_1,\dots,\xi_N)$ by e_{m-k} for short)

$$\begin{aligned}
&\sum_{k=0}^{N}(-1)^k\Big(\prod_{j=1}^{k}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{1\le i_1<\cdots<i_{N-k}\le N}e_m(\overbrace{\xi_0,\dots,\xi_0}^{k\text{-times}},\xi_{i_1},\dots,\xi_{i_{N-k}})\\
&=\sum_{l=0}^{N}(-1)^l\Big(\prod_{j=1}^{l}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{i_1>\cdots>i_{N-l}}\sum_{k=0}^{\min(l,m)}\sum_{j_1<\cdots<j_{m-k}}\xi_{i_{j_1}}\cdots\xi_{i_{j_{m-k}}}\binom{l}{k}\xi_0^k\\
&=\sum_{l=0}^{N}(-1)^l\Big(\prod_{j=1}^{l}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{k=0}^{\min(l,m)}\sum_{i_1>\cdots>i_{m-k}}\binom{N-m+k}{l}\binom{l}{k}\xi_{i_1}\cdots\xi_{i_{m-k}}\xi_0^k\\
&=\sum_{k=0}^{m}(-1)^k\Big(\prod_{j=1}^{k}\frac{j(K-1)+1}{j(K-1)}\Big)\\
&\qquad\cdot\sum_{l=k}^{N}(-1)^{l-k}\binom{N-m+k}{l}\binom{l}{k}\prod_{j=k+1}^{l}\Big(1+\frac{1}{j(K-1)}\Big)e_{m-k}\xi_0^k.
\end{aligned}$$

Applying Lemma 3.6 for $q=1/(K-1)$, we obtain the identity of the lemma. □

Setting $\xi_i = \rho_i$ in Lemma 3.7 yields the following identities for the elements W'_m ($m = 1, \ldots, N$):

$$W'_m = \sum_{k=0}^{N}(-1)^k\Big(\prod_{j=1}^{k}\frac{j(K-1)+1}{j(K-1)}\Big)\sum_{i_1<\cdots<i_{N-k}} e_m(\overbrace{\rho_0,\ldots,\rho_0}^{k\text{-times}},\rho_{i_1},\ldots,\rho_{i_{N-k}})\mathbf{1}$$
$$= \Big(\prod_{j=1}^{N-m}\frac{j(K-1)-K}{j(K-1)}\Big)\sum_{k=0}^{m}(-1)^k\Big(\prod_{j=1}^{k}\frac{j(K-1)+1}{j(K-1)}\Big)e_{m-k}(\rho_1,\ldots,\rho_N)\rho_0^k\mathbf{1}.$$

Definition 3.8. *For $m = 1, \ldots, N$, set*

$$W''_m = (-1)^m\Big(\prod_{j=1}^{N-m}\frac{j(K-1)}{j(K-1)-K}\Big)W'_m \tag{5}$$
$$= \sum_{k=0}^{m}(-1)^{m+k}\Big(\prod_{j=1}^{k}\frac{j(K-1)+1}{j(K-1)}\Big)\cdot e_{m-k}(\rho_1,\ldots,\rho_N)\rho_0^k\mathbf{1} \in \mathcal{W}^K_{\mathbb{C}[K]}.$$

Note that $(K-1)^{-1}\rho_0 = \partial + Y_{(-1)}$ is an operator defined over $\mathbb{C}[K]$, and hence W''_m is a well-defined element in $\mathcal{W}^K_{\mathbb{C}[K]}$. Then, we define elements $W_m \in \mathcal{W}^K_{\mathbb{C}[K]}$ for $m = 1, \ldots, N$ inductively as follows:

$$W_m = W''_m - \sum_{k=0}^{m-1}(-1)^{m-k}\binom{N-k}{m-k}W_{k(-1)}((K-1)\partial + H_{(-1)})^{m-k}\mathbf{1}. \tag{6}$$

In particular, $W_0 = \mathbf{1}$, $W_1 = 0$.

In [10, Lemma 2.3.5], a conformal vector ω of the vertex algebra $\mathcal{W}^K_{\mathbb{C}[K]} \otimes \mathbb{C}[K, K^{-1}]$ is explicitly given over $\mathbb{C}[K, K^{-1}]$. By direct calculation, we have the following relations between W_2, H and ω.

Proposition 3.9. *We have the following identities between the elements W_2 and H, ω;*

$$\omega = -\frac{N}{K}W_2 - \frac{N}{2}\partial H$$

in $\mathcal{W}^K_{\mathbb{C}[K]} \otimes \mathbb{C}[K, K^{-1}]$.

In the rest of this section, we show that the $N+1$ elements E, F, H and W_m for $m = 2, \ldots, N-1$ strongly generate the subregular $\mathcal{W}$-algebra $\mathcal{W}^{K_0} = \mathcal{W}^K_{\mathbb{C}[K]} \otimes \mathbb{C}_{K_0}$ for $K_0 \in \mathbb{C}$.

For a vertex algebra V, let $\overline{A}(V) = V/C_2(V)$ be Zhu's C_2 Poisson algebra of V, where $C_2(V) = V_{(-2)}V$. For an element $a \in V$, we denote its image in $\overline{A}(V)$ by $\overline{a} \in \overline{A}(V)$.

Lemma 3.10. *We have the identity*

$$e_m(\overline{X}_1 - \overline{H}, \ldots, \overline{X}_N - \overline{H}) = \sum_{k=0}^{m}(-1)^{m-k}\binom{N-k}{m-k}e_k(\overline{X}_1, \ldots, \overline{X}_N)\overline{H}^{m-k}.$$

Proof. Comparing the coefficients of u^{N-m} of the expansions of the generating functions

$$(u + \overline{X}_N - \overline{H})\cdots(u + \overline{X}_1 - \overline{H}) = \sum_{m=0}^{N} e_m(\overline{X}_1 - \overline{H}, \ldots, \overline{X}_N - \overline{H})u^{N-m}$$

with

$$
\begin{aligned}
(u+\overline{X}_N-\overline{H})\cdots(u+\overline{X}_1-\overline{H}) &= \sum_{k=0}^{N} e_k(\overline{X}_1,\dots,\overline{X}_N)(u-\overline{H})^{N-k}\\
&= \sum_{k=0}^{N} e_k(\overline{X}_1,\dots,\overline{X}_N)\sum_{m=0}^{N-k}\binom{N-k}{m-k}(-\overline{H})^{m-k}u^{N-m}\\
&= \sum_{m=0}^{N}\sum_{k=0}^{m}(-1)^{m-k}\binom{N-k}{m-k}e_k(\overline{X}_1,\dots,\overline{X}_N)\overline{H}^{m-k}u^{N-m},
\end{aligned}
$$

we obtain the identity of the assertion. □

Lemma 3.11. *For arbitrary $K_0\in\mathbb{C}$, we have $\overline{W}_m = e_m(\overline{X}_1,\dots,\overline{X}_N)$ in $\overline{A}(\mathcal{V}^{K_0})$ for $m=1,\dots,N$.*

Proof. Since $Y = e^{\int Y}_{(-2)}e^{-\int Y} \equiv 0$ modulo $C_2(\mathcal{V}^{K_0})$, it follows that $(K-1)^{-1}\rho_0 = \partial + Y_{(-1)} \equiv 0$ modulo $C_2(\mathcal{V}^{K_0})$. Using (5) and Lemma 3.10, we have

$$
\begin{aligned}
&\overline{W}''_m = (-1)^m e_m(\rho_1,\dots,\rho_N)\\
&= e_m(\overline{X}_1-\overline{H},\dots,\overline{X}_N-\overline{H}) = \sum_{k=0}^{m}(-1)^{m-k}\binom{N-k}{m-k}e_k(\overline{X}_1,\dots,\overline{X}_N)\overline{H}^{m-k}.
\end{aligned}
$$

Then, the assertion of the lemma follows from the definition (6) of W_m by induction on m. □

Lemma 3.12. *For arbitrary $K_0\in\mathbb{C}$, the elements $\overline{H}$, $\overline{W}_2$, …, $\overline{W}_N \in \overline{A}(\mathcal{W}^{K_0})$ are algebraically independent over the field $\mathbb{C}$.*

Proof. To show that $\overline{H}$, $\overline{W}_2$, …, $\overline{W}_N$ are algebraically independent in $\overline{A}(\mathcal{W}^{K_0})$, it is enough to show that they are algebraically independent in $\overline{A}(\mathcal{V}^{K_0})$, since we have $C_2(\mathcal{W}^{K_0})\subset C_2(\mathcal{V}^{K_0})$. Below we write $\overline{e}_m = e_m(\overline{X}_1,\dots,\overline{X}_N)$ for short. First, observe that non-zero elements in the lattice part of $\mathcal{V}_R^K$ modulo $C_2(\mathcal{V}^{K_0})$ form linear combinations of $e^{m\int Y}$ for $m\in\mathbb{Z}$. Thus, $\overline{Q}$, $\overline{A}_1$, …, $\overline{A}_{N-1}$ are linearly independent in $\overline{A}(\mathcal{V}^{K_0})$, while $Y = e^{\int Y}_{(-2)}e^{-\int Y} \equiv 0$ modulo $C_2(\mathcal{V}^{K_0})$. Since $\overline{X}_i \in \bigoplus_{j=1}^{N-1}\mathbb{C}\overline{A}_j$ for all $i=1,\dots,N$ and $\overline{H}=\overline{X}_0\notin\bigoplus_{j=1}^{N-1}\mathbb{C}\overline{A}_j$, $\overline{H}$ is algebraically independent of $\overline{W}_2=\overline{e}_2$, …, $\overline{W}_N=\overline{e}_N$.

We identify the vector space $\bigoplus_{j=1}^{N-1}\mathbb{C}\overline{A}_j$ with the Cartan subalgebra

$$
\mathfrak{h} = \left\{ \sum_{i=1}^{N} c_i\varepsilon_i \in \bigoplus_{i=1}^{N}\mathbb{C}\varepsilon_i \;\middle|\; c_1+\cdots+c_N = 0\right\} \simeq \mathbb{C}^{N-1}
$$

of $\mathfrak{sl}_N$ by the standard way; $\overline{A}_j = \varepsilon_j - \varepsilon_{j+1}$ $(j=1,\dots,N-1)$. Under this identification, we have $\overline{X}_i = \varepsilon_i - (1/N)\sum_{j=1}^{N}\varepsilon_j$ for $i=1,\dots,N$. Then, the symmetric polynomials $\overline{e}_2$, …, $\overline{e}_N$ are algebraically independent and we have $\mathbb{C}[\mathfrak{h}]^{\mathfrak{S}_N} = \mathbb{C}[\overline{e}_2,\dots,\overline{e}_N]$, while $\overline{e}_1 = 0$ by the classical fact on the Weyl-group-invariant subalgebra $\mathbb{C}[\mathfrak{h}]^{\mathfrak{S}_N}$. Thus, we have the assertion of the lemma. □

Proposition 3.13. *The Poisson center of C_2 Poisson algebra $\overline{A}(\mathcal{W}^{K_0})$ is generated by $\overline{W}_2$, …, $\overline{W}_N$.*

Proof. We use the same notations $\overline{e}_m = e_m(\overline{X}_1,\dots,\overline{X}_N)$ as in the proof of Lemma 3.12. Since $(X_i,Y)=0$ for $i=1,\dots,N$, it is easy to check that $\{\overline{X}_i,\overline{E}\} = \{\overline{X}_i,\overline{F}\} = 0$. Thus, $\overline{e}_2$, …, $\overline{e}_N$ are Poisson central elements in $\overline{A}(\mathcal{V}^{K_0})$. By [16, Lemma 6.12], $\mu_{\beta\gamma}\circ\mu_W\circ\mu : \mathcal{W}^{K_0} \longrightarrow \mathcal{V}^{K_0}$ induces an embedding $\overline{A}(\mathcal{W}^{K_0}) \hookrightarrow \overline{A}(\mathcal{V}^{K_0})$, and thus

$\overline{e}_2, \ldots, \overline{e}_N$ are also Poisson central elements in $\overline{A}(\mathcal{W}^{K_0})$. Using results of [7], it follows that the Poisson center of $\overline{A}(\mathcal{W}^{K_0})$ is isomorphic to $\mathbb{C}[\mathfrak{h}]^{\mathfrak{S}_N}$. Then $\overline{X}_2, \ldots, \overline{X}_N$ are algebraically independent by Lemma 3.12. Thus, the Poisson center of $\overline{A}(\mathcal{W}^{K_0})$ is $\mathbb{C}[\overline{W}_2, \ldots, \overline{W}_N] \simeq \mathbb{C}[\mathfrak{h}]^{\mathfrak{S}_N}$. □

It is easy to check that the elements E, H, W_2, W_3, …, W_{N-1}, F have conformal weights 1, 1, 2, 3, …, $N-1$, $N-1$ respectively. Note that, for arbitrary $K_0 \in \mathbb{C}$, the vertex algebra $\mathcal{W}^{K_0}$ is of type $\mathcal{W}(1,1,2,3,\ldots,N-1,N-1)$, i.e. $\mathcal{W}^{K_0}$ has $N+1$ strong generators of conformal weight 1, 1, 2, 3, …, $N-1$, $N-1$.

Theorem 3.14. *For arbitrary $K_0 \in \mathbb{C}$, the elements E, H, W_2, …, W_{N-1}, F strongly generate the vertex algebra $\mathcal{W}^{K_0}$.*

Proof. The vertex algebra $V = \mathcal{W}^{K_0}$ is decomposed as $V = \bigoplus_{d \geq 0} V_d$ where V_d is the subspace of conformal weight d. Since V is of type $\mathcal{W}(1,1,2,\ldots,N-1,N-1)$, V_1 is two-dimensional. Now that we know E and H are elements linearly independent in V_1, we have $V_1 = \mathbb{C}E \oplus \mathbb{C}H$.

By induction on d, we show that

$$V_d \subset \mathrm{Span}\{a_{1(-n_1)} \cdots a_{k(-n_k)}\mathbf{1} \,|\, a_i = E, H, W_2, \ldots, W_d, n_i \geq 1\} \tag{7}$$

for $d = 2$, …, $N-1$. Assume that (7) holds for $V_{d'}$ with $d' \leq d-1$. Set

$$U_d = V_d \cap \mathrm{Span}\{a_{1(-n_1)} \cdots a_{k(-n_k)}\mathbf{1} \,|\, a_i = E, H, W_2, \ldots, W_{d-1}, n_i \geq 1\}.$$

Since V has exactly one strong generator of conformal weight d, U_d is codimension one in V_d. We show that $W_d \notin U_d$. Indeed, assume that we have an identity

$$W_d = \sum_p c^{(p)} a^{(p)}_{1(-n_1)} \cdots a^{(p)}_{k_p(-n_{k_p})}\mathbf{1}, \tag{8}$$

where $c^{(p)} \in \mathbb{C}$, $a^{(p)}_i = E, H, W_2, \ldots, W_{d-1}$ and $n_i \geq 1$. Terms $a^{(p)}_{1(-n_1)} \cdots a^{(p)}_{k_p(-n_{k_p})}\mathbf{1}$ containing $E_{(-n)}$ $(n \geq 1)$ have positive $H_{(0)}$-eigenvalues while $H_{(0)}W_m = 0$ for all m and $H_{(0)}H = 0$. By using decomposition of U_d into $H_{(0)}$-eigenspaces, we may assume that the identity (8) holds for $a^{(p)}_i = H$, W_2, …, W_{d-1}. Taking modulo $C_2(V)$ in (8), we have an identity $\overline{W}_d = \sum_p c^{(p)} \overline{a}^{(p)}_{1(-n_1)} \cdots \overline{a}^{(p)}_{k_p(-n_{k_p})}\mathbf{1}$ in the algebra $\overline{A}(V)$, which contradicts Lemma 3.12. Hence we have $W_d \notin U_d$. Thus, we have $V_d = U_d \oplus \mathbb{C}W_d$ and the induction completes.

Similarly to the above, setting

$$U_{N-1} = V_{N-1} \cap \mathrm{Span}\{a_{1(-n_1)} \cdots a_{k(-n_k)}\mathbf{1} \,|\, a_i = E, H, W_2, \ldots, W_{N-1}, n_i \geq 1\},$$

we have $\dim V_{N-1}/U_{N-1} = 2$. Since the element $\overline{W}_{N-1}$ is algebraically independent of $\overline{H}$, $\overline{W}_2$, …, $\overline{W}_{N-2}$ in $\overline{A}(V)$ by Lemma 3.12, we have $W_{N-1} \notin U_{N-1}$. Note that the element F has weight -1 with respect to the action of $H_{(0)}$, and thus we have $F \notin U_{N-1} \oplus \mathbb{C}W_{N-1}$. Therefore we have $V_{N-1} = U_{N-1} \oplus \mathbb{C}W_{N-1} \oplus \mathbb{C}F$.

Since V have no strong generator with conformal weight bigger than $N-1$, we have

$$V = \mathrm{Span}\{a_{1(-n_1)} \cdots a_{k(-n_k)}\mathbf{1} \,|\, a_i = E, H, W_2, \ldots, W_{N-1}, F, n_i \geq 1\}.$$

Thus, E, H, W_2, …, W_{N-1}, F strongly generate V. □

4. Structure of the subregular $\mathcal{W}$-algebra at critical level

In this section, we consider the strong generators E, H, W_2, …, W_{N-1}, F of the subregular $\mathcal{W}$-algebra at critical level ($K = 0$), and study the OPEs between these generators. In particular, we give a proof of the Adamović's conjecture. Throughout this section, we specialize K to 0, and consider the elements E, F, H, W_2, …, W_N in the vertex algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}) = \mathcal{W}^0 = \mathcal{W}^K_{\mathbb{C}[K]} \otimes \mathbb{C}_0$.

The following lemma is essentially due to Molev. See [21, Proposition 12.4.4].

Lemma 4.1. *For $m = 1, \dots, N$, we have the following identity*

$$e_m(\partial - H_{(-1)} + X_{1(-1)}, \dots, \partial - H_{(-1)} + X_{N(-1)}) = \sum_{k=0}^{m} \binom{N-k}{m-k} W_{k(-1)}(\partial - H_{(-1)})^{m-k}.$$

Proof. Define $\zeta_m \in \mathbb{C}[X_{i(-n)} \mid i = 1, \dots, N, n \geq 1]$ by

$$\sum_{m=0}^{N} \zeta_m \partial^m = (\partial + X_{N(-1)}) \cdots (\partial + X_{1(-1)}).$$

Let u be an indeterminate. Replacing ∂ by $u + \partial$ yields the identity

$$\begin{aligned}(u + \partial + X_{N(-1)}) \cdots (u + \partial + X_{1(-1)}) &= \sum_{k=0}^{N} \zeta_k (u + \partial)^{N-k} \\ &= \sum_{m=0}^{N} \sum_{k=0}^{m} \binom{N-k}{m-k} \zeta_k \partial^{m-k} u^{N-m}.\end{aligned}$$

On the other hand, the LHS is clearly equal to

$$\sum_{m=0}^{N} e_m(\partial + X_{1(-1)}, \dots, \partial + X_{N(-1)}) u^{N-m},$$

and thus we have $\zeta_m \mathbf{1} = e_m(\partial + X_{1(-1)}, \dots, \partial + X_{N(-1)})\mathbf{1} = W_m$. Since $X_1, \dots, X_N$ are central, it follows that $\zeta_m = W_{m(-1)}$.

Since $H_{(-1)}$ commutes with $X_{i(-1)}$ for $i = 1, \dots, N$, one can replace ∂ by $\partial - H_{(-1)}$ in the above identity. Thus, we obtain that

$$e_m(\partial - H_{(-1)} + X_{1(-1)}, \dots, \partial - H_{(-1)} + X_{N(-1)}) = \sum_{m=0}^{N} \sum_{k=0}^{m} \binom{N-k}{m-k} W_{k(-1)}(\partial - H_{(-1)})^{m-k}$$

as the coefficient of u^{N-m} for $m = 1, \dots, N$. $\square$

When $K = 0$, by Lemma 4.1, we have

$$\begin{aligned}W''_m &= (-1)^m e_m(\rho_1, \dots, \rho_N)\mathbf{1} = e_m(\partial - H_{(-1)} + X_{1(-1)}, \dots, \partial - H_{(-1)} + X_{N(-1)})\mathbf{1} \\ &= \sum_{k=0}^{m} \binom{N-k}{m-k} e_k(\partial + X_{1(-1)}, \dots, \partial + X_{N(-1)})(\partial - H_{(-1)})^{m-k}\mathbf{1}.\end{aligned}$$

Then, for each W_m defined by (6), one can easily check by induction on m that

$$W_m = e_m(\partial + X_{1(-1)}, \dots, \partial + X_{N(-1)})\mathbf{1}$$

for $m = 0, 1, \dots, N$. Recall that $W_0 = \mathbf{1}$ and $W_1 = 0$. Since $X_i \in \bigoplus_{j=1}^{N-1} \mathbb{C}A_j$ for $i = 1, \dots, N$ and $(A_j, —) = 0$ for $j = 1, \dots, N-1$, the element W_m is central for $m = 2, \dots, N$.

Proposition 4.2. *The elements $W_2, \dots, W_N$ strongly generate the center of the vertex algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$.*

Proof. By [6, Theorem 12.1] (which is originally stated in [2, Theorem 1.1]), the center of the vertex algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ coincides with the center of the universal affine vertex algebra $V^{-N}(\mathfrak{sl}_N)$, and it is $\mathbb{Z}_{\geq 0}$-graded. Hence, the center of $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ is the commutative vertex algebra of type $\mathcal{W}(2, 3, \dots, N)$. Note that the elements $\overline{W}_2, \ldots, \overline{W}_N$ are algebraically independent in the C_2 Poisson algebra by Lemma 3.12. Then one can apply the same argument in the proof of Theorem 3.14 to the elements $W_2, \ldots, W_N$ of the center, whence the assertion. $\square$

Note that we know the OPEs

(9)
$$H(z)E(w) \sim \frac{1}{z-w}E(w), \quad H(z)F(w) \sim \frac{-1}{z-w}F(w), \quad H(z)H(w) \sim \frac{-1}{(z-w)^2},$$

and $E(z)E(W) \sim F(z)F(w) \sim 0$. To describe complete structure of the vertex algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ in terms of OPE, we need to compute the OPE between E and F. First, by direct computation, we have the following lemma.

Lemma 4.3. *We have the commutation relation* $[e^{\int Y}_{(m)}, \rho_i] = (-m-1)e^{\int Y}_{(m-1)}$ *for* $i = 1, \ldots, N$.

Now we describe the OPE between E and F in terms of our strong generators.

Theorem 4.4. *We have the following OPE:*

$$E(z)F(w) \sim (-1)^{N+1}\sum_{n=1}^{N-1}\frac{n!}{(z-w)^n}\sum_{m=0}^{N-n}\binom{N-m}{n}\Big(W_{m(-1)}(\partial - H_{(-1)})^{N-n-m}\mathbf{1}\Big)(w).$$

Proof. For $m \leq N+1$, using Lemma 4.3 together with the facts that $e^{\int Y}_{(n)}e^{-\int Y} = 0$ for $n \geq 0$ and $e^{\int Y}_{(-1)}$ commutes with ρ_i $(i = 1, \ldots N)$, we have

$$\begin{aligned}
E_{(N-m)}F &= -e^{\int Y}_{(N-m)}\rho_N \cdots \rho_1 e^{-\int Y}\\
&= -\rho_N \cdots \rho_1 e^{\int Y}_{(N-m)}e^{-\int Y} - \cdots - \Big(\prod_{j=1}^{N-m}(-j-1)\Big)\sum_{i_1 > \cdots > i_m}\rho_{i_1}\cdots\rho_{i_m}e^{\int Y}_{(0)}e^{-\int Y}\\
&\qquad + \Big(\prod_{j=1}^{N-m}(-j-1)\Big)\sum_{i_1 > \cdots > i_{m-1}}\rho_{i_1}\cdots\rho_{i_{m-1}}e^{\int Y}_{(-1)}e^{-\int Y}\\
&= (-1)^{N+1}(N-m+1)!e_{m-1}(\partial - H_{(-1)} + X_{1(-1)}, \ldots, \partial - H_{(-1)} + X_{N(-1)})\mathbf{1}.
\end{aligned}$$

Applying Lemma 4.1, we obtain

(10)
$$E_{(N-m)}F = (-1)^{N+1}(N-m+1)!\sum_{k=0}^{m-1}\binom{N-k}{m-k-1}W_{k(-1)}(\partial - H_{(-1)})^{m-k-1}\mathbf{1}.$$

This immediately implies the OPE relation of the theorem. $\square$

Using the strong generators E, F, H and W_m $(m = 2, \ldots, N-1)$, we will give the explicit algebraic structure of both of the Zhu's C_2-Poisson algebra and Zhu algebra of $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$.

Let $V = \bigoplus_{\Delta \geq 0} V_\Delta$ be a $\mathbb{Z}_{\geq 0}$-graded vertex algebra, and we denote the degree of a homogeneous element $a \in V$ by $\Delta(a)$. For $a \in V_\Delta$ and $b \in V$, we define

$$a \circ b = \sum_{j=0}^{\Delta}\binom{\Delta}{j}a_{(j-2)}b, \qquad a * b = \sum_{j=0}^{\Delta}\binom{\Delta}{j}a_{(j-1)}b.$$

Then, the vector space $A(V) = V/(V \circ V)$ has a structure of an associative algebra by the multiplication induced by $*$, called the Zhu algebra of V [25, 13, 7]. For $V = \mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$, the Zhu algebra $A(\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})) = U(\mathfrak{sl}_N, f_{sub})$ is known as the finite $\mathcal{W}$-algebra associated with $\mathfrak{sl}_N$ and f_{sub} by the result of De Sole and Kac in [7], and in particular it does not depend on the level $K_0 - N$. See also

[3]. Moreover, in [22], Premet showed that the finite $\mathcal{W}$-algebra $U(\mathfrak{sl}_N, f_{sub})$ was isomorphic to Smith's algebra introduced by Smith in [23]. Below we describe the structure of $A(\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}))$.

Note that the vertex algebra $\mathcal{V}^0$ is $\mathbb{Z}$-graded by $\Delta(A_i) = 1$ for $i = 1, \ldots, N-1$, $\Delta(Q) = 1$, $\Delta(Y) = 1$ and $\Delta(e^{\pm \int Y}) = \pm 1$, which induces a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}) = \mathcal{W}^0 \subset \mathcal{V}^0$; i.e. $\Delta(E) = 1$, $\Delta(F) = N-1$, $\Delta(H) = 1$ and $\Delta(W_m) = m$ for $m = 2, \ldots, N$. This grading on $\mathcal{W}^0$ coincides with the conformal weight although $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ is not a vertex operator algebra.

First, consider the C_2 Poisson algebra $\overline{A} := \overline{A}(\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}))$. The elements $\overline{W}_2, \ldots, \overline{W}_N$ are Poisson central elements and algebraically independent by Proposition 3.13 and Lemma 3.12. Thus, $\overline{A}$ is a Poisson algebra over $\mathbb{C}[\overline{W}_2, \ldots, \overline{W}_{N-1}]$. By (10) for $m = N+1$, we have

$$\overline{EF} = \sum_{k=0}^{N} (-1)^{k+1} \overline{W}_k \overline{H}^{N-k}.$$

Together with the results of [7] (or [3], in which the principal cases are only concerned, but one can easily adapt to the general cases), the Poisson algebra $\overline{A}$ is the associated graded algebra of $U(\mathfrak{sl}_N, f_{sub})$ with respect to the Kazhdan filtration. In particular, $\overline{A}$ is the coordinate ring of the Slodowy slice $\mathbb{S}$ in $\mathfrak{sl}_N$ associated with f_{sub}, which is known as the simultaneous deformation of the Kleinian singularity of type A_{N-1}. These fact gives an isomorphism of commutative algebras

$$\overline{A} = \mathbb{C}[\overline{E}, \overline{F}, \overline{H}, \overline{W}_m \,|\, m = 2, \ldots, N-1] \,/\, \big(\overline{EF} - \sum_{k=0}^{N} (-1)^{k+1} \overline{W}_k \overline{H}^{N-k}\big),$$

and the subalgebra $\mathbb{C}[\overline{W}_2, \ldots, \overline{W}_N]$ is the Poisson center of $\overline{A}$. It then follows from (10) that the Poisson brackets between these elements are given by

$$\{\overline{H}, \overline{E}\} = \overline{E}, \quad \{\overline{H}, \overline{F}\} = -\overline{F}, \quad \{\overline{E}, \overline{F}\} = \sum_{k=0}^{N} (-1)^{k+1} (k+1) \overline{W}_k \overline{H}^{N-k-1}.$$

Let $A := A(\mathcal{W}^0)$ be the Zhu algebra of $\mathcal{W}^0 = \mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$, and $\widetilde{W} = W + (\mathcal{W}^0 \circ \mathcal{W}^0)$ be the image of W under the canonical projection $\mathcal{W}^0 \twoheadrightarrow A = \mathcal{W}^0/(\mathcal{W}^0 \circ \mathcal{W}^0)$ for all $W \in \mathcal{W}^0$. An easy consequence of Theorem 3.14 and Proposition 4.2 is that A is generated by $\widetilde{E}$, $\widetilde{F}$, $\widetilde{H}$ and $\widetilde{W}_m$ ($m = 2, \ldots, N-1$), and the center of A coincides with $\mathbb{C}[\widetilde{W}_2, \ldots, \widetilde{W}_N]$. By (9), we have that $[\widetilde{H}, \widetilde{E}] = \widetilde{E}$ and $[\widetilde{H}, \widetilde{F}] = -\widetilde{F}$, and moreover, using (10) and the skew-symmetry $F_{(n)}E = \sum_{j\geq 0} (-1)^{n+j-1} \partial^j (E_{(n+j)}F)/j!$, $[\widetilde{E}, \widetilde{F}]$ is a polynomial in $\widetilde{H}$ of degree $N-1$ with coefficients in $\mathbb{C}[\widetilde{W}_2, \ldots, \widetilde{W}_{N-1}]$. Thus, the Zhu algebra A is isomorphic to Smith's algebra [23] over $\mathbb{C}[\widetilde{W}_2, \ldots, \widetilde{W}_{N-1}]$, which recovers results by Premet [22, Theorem 7.10].

In [1], Adamović conjectured a construction of the vertex algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ by using fermionic fields, which we will recall below.

Set

$$\alpha = Q + \frac{N-1}{N} A_1 + \cdots + \frac{1}{N} A_{N-1}, \qquad \beta = Y - \alpha,$$

elements of the vertex algebra $\mathcal{V}^0$. Note that we have $(\alpha, \alpha) = 1$, $(\beta, \beta) = -1$, $(\alpha, \beta) = 0$. We consider the fermionic vertex operators $\Psi^{\pm}(z) := e^{\pm \int \alpha}(z)$ and $e^{\pm \int \beta}(z)$. Since $H = -\beta$ and $e^{-\int Y} = \Psi^{-}_{(-1)} e^{-\int \beta}$, we have $(\partial - H_{(-1)})^n \, e^{-\int Y} =$

$(\partial^n\Psi^-)_{(-1)}e^{-\beta}$ for $n \geq 0$. Then, we have

$$\begin{aligned}F &= -\rho_N \cdots \rho_1 e^{-\int Y}\\&= (-1)^{N+1}(\partial - H_{(-1)} + X_{N(-1)}) \cdots (\partial - H_{(-1)} + X_{1(-1)})e^{-\int Y}\\&= (-1)^{N+1}\sum_{m=0}^{N} W_{m(-1)}(\partial - H_{(-1)})^{N-m}e^{-\int Y}\\&= (-1)^{N+1}\sum_{m=0}^{N} W_{m(-1)}(\partial^{N-m}\Psi^-)_{(-1)}e^{-\beta}\end{aligned}$$

since $H_{(-1)}$ commutes with $X_{i(-1)}$ for $i = 1, \ldots, N$. Therefore, we obtain the following realization of the vertex algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ as conjectured in [1]:

Theorem 4.5 (Adamović's conjecture [1]). *For $N \geq 2$, the subregular $\mathcal{W}$-algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ of type A_{N-1} is isomorphic to the vertex algebra strongly generated by the following fields:*

$$\begin{gathered}{}^{\circ}_{\circ} e^{\int\beta}(z)\Psi^+(z){}^{\circ}_{\circ}, \quad H(z), \quad W_m(z) \quad (m = 2, \ldots, N-1),\\ \textit{and} \ \sum_{m=0}^{N} {}^{\circ}_{\circ} W_m(z)e^{-\int\beta}(z)\partial^{N-m}\Psi^-(z){}^{\circ}_{\circ}.\end{gathered}$$

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(N.G.) Department of Mathematical and Statistical Sciences,University of Alberta, Edmonton, AB T6G 2G1, Canada.

Email address: genra@ualberta.ca

(T.K.) Department of Mathematics, Faculty of Pure and Applied Sciences, University of Tsukuba Tsukuba, Ibaraki 305-8571, JAPAN.

Email address: kuwabara@math.tsukuba.ac.jp