Helfer, Joseph

Homotopies in Grothendieck fibrations. (English) Zbl 07229472

Theory Appl. Categ. 35, 1312-1378 (2020).

The notion of *Grothendieck fibration* as well as the essentially equivalent notion of *pseudo-functor* was introduced in [*A. Grothendieck*, in: Sem. Bourbaki 12 (1959/60), No. 190, 29 p. (1960; Zbl 0229.14007)] so as to formulate the notion of *descent*. Later, Lawvere introduced fibrations into categorical logic with his theory of *hyperdoctrines* [*F. W. Lawvere*, Repr. Theory Appl. Categ. 2006, No. 16, 1–16 (2006; Zbl 1114.18002); Dialectica 23, 281–296 (1969; Zbl 0341.18002)]. This paper aims to exhibit a naturally occurring 2-categorical structure on the base category of any Grothendieck fibration abiding by certain assumptions.

The paper, consisting of 15 sections, is divided into four parts. The first part is composed of an introduction and §1, which is a cursory review on fibrations.

Part II, ranging from §2 through §8, aims to define a 2-category structure on the base category of any $\wedge$ =-fibration

$$\mathscr{C}\ \begin{matrix}\mathbf{C}\\ \downarrow\\ \mathbf{B}\end{matrix}$$

§2 defines the notion of $\mathscr{C}$-homotopy and the *vertical* composition, showing that they give rise to a category, which is shown to be a groupoid in §3. §4 defines the *horizontal* composition, which, together with the vertical composition, forms a 2-category. In §5 the author gives an alternative presentation of the 2-categorical structure in terms of internal categories. §6 is engaged in carrying out the extension of the fibration to a 1-discrete 2-fibration. It is shown in §7 that the 2-category has finite products. §8 is devoted to presenting the universal property characteristic of the 2-categorical structure.

Part III, occupying §9–§13, is concerned with some examples of $\wedge$ =-fibrations. §9 recalls the definition of the *codomain fibration*

$$\mathscr{F}(\mathbf{C})\ \begin{matrix}\mathbf{C}^{\to}\\ \downarrow\\ \mathbf{C}\end{matrix}$$

which is a $\wedge$ =-fibration precisely when $\mathbf{C}$ has finite limits. However, the resulting 2-categorical structure is trivial (§9.2). §10 recalls the definition of Quillen model categories and their basic properties. In §11 the author defines the filtration

$$\mathscr{H}o\mathscr{F}(\mathbf{C})\ \begin{matrix}\mathbf{Ho}(\mathbf{C}^{\to})\\ \downarrow\\ \mathbf{C}\end{matrix}$$

which is shown in §12 to be a $\wedge$-fibration. §13 shows that its restriction to the fibrant objects, denoted by $\mathbf{C}_f$ renders a $\wedge$ =-fibration.

Part IV, taking §14 and §15, relates the 2-categorical structure on a model category arising from Parts II and III to the standard 2-categorical structure on it. It is suggested but not proved that any functor from the 1-category $\mathbf{C}_{cf}$ to a 2-category $\mathbf{D}$ taking every weak equivalence to an equivalence in $\mathbf{D}$ extends uniquely to the 2-categorical structure on $\mathbf{C}_{cf}$.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18D30 Fibered categories
55U35 Abstract and axiomatic homotopy theory in algebraic topology

Keywords:

Grothendieck fibrations; hyperdoctrine; 2-category

Full Text: Link

References:

[1] Steve Awodey and Michael A. Warren. Homotopy theoretic models of identity types. Math. Proc. Cambridge Philos. Soc., 146(1):45-55, 2009. · Zbl 1205.03065

[2] Francis Borceux.Handbook of categorical algebra. 1, volume 50 ofEncyclopedia of Mathematics and its Applications.Cambridge University Press, Cambridge, 1994. Basic category theory. · Zbl 0911.18001

[3] Pierre Cagne. Towards a homotopical algebra of dependent types, 2018. Universit¨e Sorbonne Paris Cit¨e.

[4] Michael Cole. Mixing model structures.Topology Appl., 153(7):1016-1032, 2006. · Zbl 1094.55015

[5] Jean Giraud.Cohomologie non ab´elienne.Springer-Verlag, Berlin-New York, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 179.

[6] Alexander Grothendieck. Technique de descente et th´eor‘emes d'existence en g´eom´etrie alg´ebrique. I. G´en´eralit´es. Descente par morphismes fid‘element plats. InS´eminaire Bourbaki, Vol. 5, pages Exp. No. 190, 299-327. Soc. Math. France, Paris, 1995.

[7] Joseph Helfer. First-order homotopical logic. 2019. Preprint: arXiv:1908.08944.

[8] Philip S. Hirschhorn.Model categories and their localizations, volume 99 ofMathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003.

[9] Mark Hovey.Model categories, volume 63 ofMathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999.

[10] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. InTwenty-five years of constructive type theory (Venice, 1995), volume 36 ofOxford Logic Guides, pages 83-111. Oxford Univ. Press, New York, 1998. A slightly different version is available athttps://web.archive.org/web/20170705123200/https:// www2.mathematik.tu-darmstadt.de/~streicher/venedig.ps.gz. · Zbl 0930.03089

[11] Bart Jacobs.Categorical logic and type theory, volume 141 ofStudies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1999. · Zbl 0911.03001

[12] Andr´e Joyal. The theory of quasi-categories and its applications. 2008. Unpublished manuscript. Avaialble athttps://web.archive.org/web/ //mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf.

[13] Chris Kapulkin and Peter Lefanu Lumsdaine.The simplicial model of univalent foundations (after voevodsky). 2012. Preprint: arXiv:1211.2851.

[14] Michael Lambert.An elementary account of flat 2-functors, 2019.PhD thesis, Dalhousie University.

[15] F. William Lawvere.Equality in hyperdoctrines and comprehension schema as an adjoint functor. InApplications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), pages 1-14. Amer. Math. Soc., Providence, R.I., 1970.

[16] F. William Lawvere. Adjointness in foundations.Repr. Theory Appl. Categ., (16):1- 16, 2006. Reprinted from Dialectica23 (1969). · Zbl 1114.18002

[17] M. Makkai. The fibrational formulation of intuitionistic predicate logic I: completeness according to G¨odel, Kripke, and L¨auchli. I.Notre Dame J. Formal Logic, 34(3):334- 377, 1993. · Zbl 0808.03049

[18] M Makkai.First order logic with dependent sorts,with applications to category theory.1995.Unpublished manuscript.Avaialble at https://web.archive.org/web/20171116044204/http://www.math.mcgill. ca/makkai/folds/foldsinpdf/FOLDS.pdf.

[19] J. P. May.A concise course in algebraic topology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. · Zbl 0923.55001

[20] Saunders Mac Lane.Categories for the working mathematician, volume 5 ofGraduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998. · Zbl 0906.18001

[21] J. P. May and K. Ponto.More concise algebraic topology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories. · Zbl 1249.55001

[22] Maria Emilia Maietti and Giuseppe Rosolini. Elementary quotient completion.Theory Appl. Categ., 27:Paper No. 17, 463, 2012. · Zbl 1288.03048

[23] James R. Munkres.Topology. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [MR0464128].

[24] Daniel G. Quillen.Homotopical algebra.Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967. · Zbl 0168.20903

[25] Agust´ı Roig. Model category structures in bifibred categories.J. Pure Appl. Algebra, 95(2):203-223, 1994. · Zbl 0811.18002

[26] Revˆetements ´etales et groupe fondamental. Lecture Notes in Mathematics, Vol. 224. Springer-Verlag, Berlin-New York, 1971. S´eminaire de G´eom´etrie Alg´ebrique du Bois Marie 1960-1961 (SGA 1), Dirig´e par Alexandre Grothendieck. Augment´e de deux expos´es de M. Raynaud.

[27] Alexandru Emil Stanculescu.Bifibrations and weak factorisation systems.Appl. Categ. Structures, 20(1):19-30, 2012. · Zbl 1252.18020

[28] Michael A Warren.