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Homotopy linear algebra. (English) Zbl 06854604

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This paper was motivated by *incidence algebras* and *Möbius transformations.* A Möbius inversion formula is an algebraic identity in the incidence algebra [*P. Leroux*, Cah. Topologie Géom. Différ. Catégoriques 16, 280–282 (1976; Zbl 0364.18001)], and hence an equation between two linear maps. Therefore, by regarding the two linear maps as spans and establishing a bijection between the sets representing these spans, one can obtain a bijective proof. *F. W. Lawvere* and *M. Menni* [Theory Appl. Categ. 24, 221–265 (2010; Zbl 1236.18001)] have established an *objective* Möbius inversion principle for Möbius 1-categories, and this paper aims to develop homotopy linear algebra for the ∞-version of these results [*I. Gálvez-Carrillo* et al., Adv. Math. 331, 952–1015 (2018; Zbl 1403.00023); Adv. Math. 334, 544–584 (2018; Zbl 1403.18017)]. Indeed, the authors have written a very long paper ["Decomposition spaces, incidence algebras and Möbius inversion", Preprint, `arXiv:1404.3202`], which was divided into six papers. This paper and the three papers mentioned above are divided ones.

J. C. Baez and *J. Dolan* [in: Mathematics unlimited – 2001 and beyond. Berlin: Springer. 29–50 (2001; Zbl 1004.18001)] discovered that the theory of species is to be enhanced by considering groupoid-valued species in place of set-valued species, showing that the exponential generating function corresponding to a species is literally the cardinality of the associated analytic functor taken with groupoid coefficients rather than set coefficients, and also showing how the annihilation and creation operators in Fock space are to be given an objective combinatorial interpretation within this setting.

J. C. Baez et al. [Theory Appl. Categ. 24, 489–553 (2010; Zbl 1229.18003)] developed in detail the basic aspects of linear algebra over groupoids under the name of *groupoidification*, showing that the symmetry factors that arise behave as expected and cancel out appropriately in the vaious manipulations, and deriving all cardinality assignments, one for each slice category, from a single global prescription, defined as a functor from groupoids and spans to vector spaces.

This paper works with coefficients in ∞-groupoids so as to incorporate more homotopy theory. This paper works with homotopy fibers and homotopy sums. Homotopy sums are left adjoint to homotopy fibers, just as, over sets, sums are left adjoint to fibers.

The motivating examples in this paper are incidence coalgebras and incidence algebras, which are naturally vector spaces and profinite-dimensional vector spaces, respectively. The fundamental fact is the classical duality between vector spaces and profinite-dimensional vector spaces.

An ∞-groupoid is called *finite* if it has finitely many components, all homotopy groups are finite, and there is an upper bound on the dimension of non-trivial homotopy groups. A morphism of ∞-groupoids is called *finite* if all its fibers are finite. Letting $\mathcal{F}$ denote the ∞-category of finite ∞-groupoids, the role of vector spaces is played by finite ∞-groupoid slices $\mathcal{F}_{/S}$, while the role of profinite-dimensional vector spaces is played by finite-presheaf ∞-category $\mathcal{F}^S$, where S is required only to be locally finite in both cases. Linear maps are given by spans of finite type, meaning

$$S \overset{p}{\leftarrow} M \overset{q}{\rightarrow} P$$

in which p is a finite map, while prolinear maps are given by spans of profinite type, meaning q is a finite map instead. The paper considers ∞-categories $\underrightarrow{\boldsymbol{lin}}$ and $\underleftarrow{\boldsymbol{lin}}$, the first of which consists of slices $\mathcal{F}_{/S}$ as objects and ∞-groupoids of finite-type spans as mapping spaces, and the second of which consists of finite-presheaf ∞-categories $\mathcal{F}^S$ as objects and ∞-groupoids of finite-type spans as mapping spaces. It is shown that they are dual. The paper introduces a global notion of cardinality such that the classical duality becomes the cardinality of the $\underrightarrow{\boldsymbol{lin}}$-$\underleftarrow{\boldsymbol{lin}}$ duality.

Finally, the theory of slices and linear functors is subsumed under the theory of polynomial functors, where a futher right adjoint enters the picture, the right adjoint to pullback. The theory of polynomial functors over ∞-categories was developed in [*D. Gepner* et al., "∞-operads as analytic monads", Preprint, arXiv:1712.06469].

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MSC:

20L05 Groupoids (i.e. small categories in which all morphisms are isomorphisms)
15A99 Basic linear algebra
46A20 Duality theory for topological vector spaces
18N50 Simplicial sets, simplicial objects
55U35 Abstract and axiomatic homotopy theory in algebraic topology

Keywords:

infinity-groupoids; homotopy cardinality; homotopy finiteness; duality; linear algebra

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