Hofmann, Karl H.; Morris, Sidney A.

The structure of compact groups. A primer for the student – a handbook for the expert. 4th revised and expanded edition. (English) Zbl 07205682

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The first edition of the book appeared in 1998 [Zbl 0919.22001], the second edition appeared in 2006 [Zbl 1139.22001], and the third edition appeared in 2013 [Zbl 1277.22001]. The reviewer is now happy to deal with its fourth edition. The great change of this fourth edition is inclusion of the Tannnaka-Krein duality, while in the past three editions the authors did not include the Tannaka duality theorem at all, because it has not contributed to the knowledge on the structure of compact groups, which is the very subject of this book.

Appendix 7 is completely new, in which the category $\mathcal{V}$ of vector spaces over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, and the category $\mathcal{W}$ of weakly complete topological $\mathbb{K}$-vector spaces are considered. A topological $\mathbb{K}$-vector space is said to be *weakly complete* iff it is, as a topological $\mathbb{K}$-vector space, isomorphic to $\mathbb{K}^J$ for some set J. It is easy to see that the dual

$$V^*=\mathrm{Hom}_{\mathcal{V}}(V,\mathbb{K})$$

of a $\mathcal{V}$-object V is a $\mathcal{W}$-object, and the dual

$$W'=\mathrm{Hom}_{\mathcal{W}}(W,\mathbb{K})$$

of a $\mathcal{W}$-object W is a $\mathcal{V}$-object with natural isomorphisms

$$V\equiv(V^*)'$$
$$W\equiv(W')^*$$

being a rather elementary duality between $\mathcal{V}$ and $\mathcal{W}$. In case of $\mathbb{K}=\mathbb{R}$, we have

$$\mathrm{Hom}_{\mathcal{V}}(V,\mathbb{K})\equiv\mathrm{Hom}(V,\mathbb{R}/\mathbb{Z})=\widehat{V}$$
$$\mathrm{Hom}_{\mathcal{W}}(W,\mathbb{K})\equiv\mathrm{Hom}(W,\mathbb{R}/\mathbb{Z})=\widehat{W}$$

so that the duality between $\mathcal{V}$ and $\mathcal{W}$ over $\mathbb{R}$ is surely a part of the Pontryagin duality.

Another noteworthy feature of $\mathcal{V}$ and $\mathcal{W}$ is that both are endowed with tensor products, with respect to which the duality preserves tensor products in the sense that

$$(V_1\otimes_{\mathcal{V}}V_2)^*\equiv V_1^*\otimes_{\mathcal{W}}V_2^*$$
$$(W_1\otimes_{\mathcal{W}}W_2)'\equiv W_1'\otimes_{\mathcal{V}}W_2'$$

Since both $\mathcal{V}$ and $\mathcal{W}$ are symmetric monoidal categories, they have algebras and Hopf algebras. There is a functor

$$A\mapsto A^{-1}$$

from the category of all weakly complete topological algebras to the category of all topological groups, where A^{-1} is the set of invertible elements in A. The functor has a left adjoint

$$G\mapsto\mathbb{K}[G]$$

where $\mathbb{K}[G]$ stands for the weakly complete group algebra of G, which is automatically weakly complete symmetric $\mathbb{K}$-Hopf algebra. We are able to characterize those real weakly complete symmetric cocommutative Hopf algebras which occur as group Hopf algebras $\mathbb{R}[G]$ for some compact group G. They are called *compactlike*. Therefore there is a categorical equivalence between the category of compact groups and the category of weakly complete compactlike real symmetric Hopf algebras.

Now comes Part 3 of Chapter 3, which is also completely new. The duality between $\mathcal{V}$ and $\mathcal{W}$ gives rise in a straightforward fashion to a duality between weakly complete cocomplete real symmetric Hopf algebras and commutative real symmetric Hopf algebras. The real symmetric Hopf algebras appearing as dual objects of the weakly complete compactlike real symmetric Hopf algebras (that is to say, $\mathbb{R}[G]$ with compact G) are called *reduced Hopf algebras* [Zbl 0131.02702]. Thus we have the duality theorem claiming that the category of compact groups is dual to the category of reduced real Hopf algebras, which is no other than the Tannaka-Hochschild duality theorem. It is also established that the dual $\mathbb{R}[G]'$ of the weakly complete real symmetric group Hopf algebra of a compact group is naturally isomorphic to the real symmetric Hopf algebra $R(G,\mathbb{R})$ of representative functions of the compact group G.

The authors' discussion in Appendix 7 and Part 3 of Chapter 3 relies heavily on [Zbl 1419.22007, Zbl 07203709].

There are some minor improvements of results in the past editions. By way of example, Theorem 6.55 is clearly formulated, claiming that, for a compact Lie group , every element of the commutator algebra $\mathfrak{g}'$ of the Lie algebra $\mathfrak{g}=\mathfrak{L}(G)$ of G is itself a commutator.

The first edition occupies 835 pages, the second edition takes 858 pages, the third edition requires 924 pages, and this fourth edition attains 1006 pages. I could not help applauding the authors' consistent enthusiasm to improve such a voluminous book. I am one of the greediest readers to anticipate the fifth edition in secret. Good Luck, Hofmann and Morris.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

22-02 Research exposition (monographs, survey articles) pertaining to topological groups
22-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to topological groups
22C05 Compact groups
22B05 General properties and structure of LCA groups
22E15 General properties and structure of real Lie groups
22E65 Infinite-dimensional Lie groups and their Lie algebras: general properties
54H11 Topological groups (topological aspects)

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