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The null identities for boundary operators in the $(2, 2p + 1)$ minimal gravity

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By using the matrix model representation, we show that correlation numbers of boundary-changing operators (BCOs) in $(2, 2p + 1)$ minimal Liouville gravity satisfy some identities, which we call the null identities. These identities enable us to express the correlation numbers of BCOs in terms of those of boundary-preserving operators. We also discuss a physical implication of the null identities as the manifestation of the boundary interaction.

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1. Introduction

Two-dimensional gravity coupled with a minimal model of CFT has been studied as a good example of well-defined quantum gravitational theories [1], and also allows a non-perturbative discrete formulation given by matrix models [2–4] (for more references, see, for example, Ref. [5]).

In this paper we follow the one-matrix model description [6,7] of the $(2, 2p + 1)$ minimal gravity on Riemann surfaces, but focus on the description in the presence of boundaries [8,9]. The boundary conditions of the minimal gravity, also referred to as FZZT branes [10], are specified by the value of the boundary cosmological constant μ_B and the Kac label $(1, m)$ of the matter Cardy state. In Ref. [9] it was shown that such boundary conditions are realized in the matrix model by introducing a generalization of the resolvent operators. The disk partition n function for the $(1, m)$ Cardy state is given by

$$F_m = -\langle \mathrm{tr} \log f_m(M)\rangle, \tag{1}$$

where $f_m(M)$ is a monic polynomial of the Hermitian matrix M with degree m, and $\langle \cdots \rangle$ stands for the expectation value of the one-matrix model. After some renormalizations, the coefficients of $f_m(M)$ are related to the sources of boundary operators, which preserves the $(1, m)$ boundary condition.

One can introduce some impurities on the boundary, which interpolate two different boundary conditions. These are called boundary-changing operators (BCOs). Between two boundary segments of $(1, m_1)$ and $(1, m_2)$ with different boundary cosmological constants, one can put a $(1, k)$ primary operator dressed by the Liouville factor $e^{\beta_k \phi}$, where $k = |m_1 - m_2| + 1, |m_1 - m_2| + 3, \ldots, m_1 + m_2 - 1$, $\beta_k = \frac{(k+1)b}{2}$, and $b^2 = 2/(2p + 1)$. It was shown in Ref. [9] that these operators are described in the

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one-matrix model as follows. One extends the disk partition functions to a 2×2 block matrix of the form

$$F_{m_1 m_2} = -\left\langle \mathrm{tr}\log \begin{pmatrix} f_{m_1}(M) & g_{m_1 m_2}(M) \\ g^{\dagger}_{m_1 m_2}(M) & f_{m_2}(M) \end{pmatrix} \right\rangle. \tag{2}$$

Here, $g_{m_1 m_2}(M)$ is a polynomial of M with degree less than $\min(m_1, m_2)$. The coefficients of $g_{m_1 m_2}(M)$ provide sources of BCOs between the $(1, m_1)$ and $(1, m_2)$ boundaries. Correlation numbers with more different boundary conditions can also be treated in a similar way by introducing more block structures. It has been shown that this formulation correctly reproduces the correlation numbers of BCOs, computed in the Liouville theory approach.

In this paper we demonstrate that the correlation numbers of BCOs satisfy some nontrivial identities, which we call null identities. The idea for deriving the null identities is the following. The perturbed partition function of Eq. (2) can be diagonalized to the form

$$F_{m_1 m_2} = -\left\langle \mathrm{tr}\log \begin{pmatrix} f'_{m_1}(M) & 0 \\ 0 & f'_{m_2}(M) \end{pmatrix} \right\rangle, \tag{3}$$

where f'_{m_1} and f'_{m_2} are new polynomials of M with degree m_1 and m_2, respectively. This shows that the sources of BCOs, which were originally encoded in the coefficients of $g_{m_1 m_2}(M)$, are actually redundant and can be absorbed into the redefinitions of the sources of the boundary-preserving operators in f_{m_1} and f_{m_2}. Thus, after the redefinitions, the partition function becomes independent of the sources of BCOs. In terms of the original parametrization, this implies that there exist differentials ∇_n $(n = 1, 2, \ldots, \min(m_1, m_2))$ such that they are given by linear combinations of the derivatives of the sources and satisfy $\nabla_n(F_{m_1 m_2}) = 0$. This is the simplest example of what we call the null identities. These identities enable us to write the correlation numbers of BCOs in terms of those of boundary-preserving operators. We will present a general derivation of the null identities and show that ∇_n can be constructed in such a way that the curvature is vanishing (i.e. $[\nabla_n, \nabla_l] = 0$). We then discuss physical implications of the identities.

This paper is organized as follows. In Sect. 2 we derive the null identities. In Sect. 3 we show some examples of the differentials ∇_n and the null identities, and discuss their physical implications. Section 4 is devoted to the conclusion and a discussion on a possible extension to cases where more than two boundary conditions are allowed. We present the case with three boundary parameters in some detail.

2. The null identities

Under the double scaling limit of the one-matrix model, insertions of the matrix M in the path integral can be replaced with insertions of a quadratic differential operator Q, which acts on the space of eigenvalues of M [11,12]. Additive and multiplicative constants appear in this replacement: $M \to \epsilon Q + c$. For insertions of polynomials of M, these constants can be absorbed into renormalizations of the coefficients of the polynomials and the overall factors of the operators. After the renormalization, the perturbed partition function takes the form

$$F_{m_1 m_2} = -\langle \mathrm{tr}\log R_{m_1 m_2}(Q)\rangle, \quad R_{m_1 m_2}(Q) = \begin{pmatrix} C_{m_1}(Q) & c(Q) \\ c(Q) & C_{m_2}(Q) \end{pmatrix}, \tag{4}$$

where $C_{m_i}(Q)$ and $c(Q)$ are polynomials obtained by renormalizing f_{m_i} and $g_{m_1m_2}$, respectively. They are written as

$$C_{m_i}(Q) = \prod_{k=1}^{m_i} \left(Q + a_k^{(i)}\right), \quad c(Q) = \sum_{n=0}^{d} c_{1+d-n} Q^n, \tag{5}$$

where $d = -1 + \min(m_1, m_2)$ and the c_k are real. The coefficients of $C_{m_i}(Q)$ and $c(Q)$ correspond to the sources of boundary-preserving and -changing operators, respectively. We will discuss this correspondence later in more detail, after we derive the null identities in the following.

By the formula $\mathrm{tr}\log R(Q) = \log \det R(Q)$, the perturbed partition function in Eq. (4) can be written as the expectation value of the logarithm of $\det(R(Q))$. As a polynomial of Q, the degree of $\det(R(Q))$ is $m = m_1 + m_2$ and it has m independent coefficients. However, the matrix $R(Q)$ has $m + d + 1$ parameters in Eq. (4). Hence, $d + 1$ parameters are redundant, and those extra coefficients can be absorbed into redefinitions of the coefficients. This implies that there exist $d + 1$ constraints on the partition function:

$$\nabla_n F_{m_1m_2} = 0, \tag{6}$$

which we refer to as null identities. Here, $n = 1, 2, \ldots, d + 1$ and ∇_n are linear differential operators given by combinations of $\left\{\frac{\partial}{\partial a_k^{(i)}}, \frac{\partial}{\partial c_n}\right\}$. The differential operators ∇_n are specified by the condition

$$\nabla_n \left(\det R_{m_1m_2}(x)\right) = 0, \tag{7}$$

where x is a formal parameter representing Q. We express $\mathrm{tr}\, R_{m_1m_2}(x)$ as

$$\det R_{m_1m_2}(x) \equiv x^m + \sum_{k=1}^{m} \zeta_k\, x^{m-k}. \tag{8}$$

The operators ∇_n specified by Eq. (7) are equivalently defined by requiring the following conditions: For all $k \in \{1, 2, \ldots, m\}$,

$$\nabla_n\, \zeta_k = 0. \tag{9}$$

A general solution to Eq. (9) can be constructed as follows. First, note that for each c_n there should exists an independent differential operator satisfying Eq. (9). Then, we put the ansatz

$$\nabla_n = \frac{\partial}{\partial c_n} + \tilde{\nabla}_n, \tag{10}$$

where $\tilde{\nabla}_n$ is a linear differential operator consisting of $\{a_k = a_k^{(1)}; a_{k+m_1} = a_k^{(2)}\}$. Specifically, $\tilde{\nabla}$ is written as

$$\tilde{\nabla}_n = \sum_{i,k} \eta_k^{(n)} \frac{\partial}{\partial a_k}, \tag{11}$$

where $\eta_k^{(n)}$ are functions of $\{a_k; c_n\}$. The coefficients $\eta_k^{(n)}$ can be determined by requiring the conditions (9). Let us introduce a set of variables,

$$\xi_i = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le m} a_{j_1} a_{j_2} \ldots a_{j_i} \,. \tag{12}$$

Then, if $\zeta_k - \xi_k$ (for all $k \in \{1, 2, \ldots, m\}$) has no explicit a_i dependence (which is always the case for a 2×2 block matrix dealing with two boundary parameters), the differential operator is given by

$$\nabla_n = \frac{\partial}{\partial c_n} - \sum_{k=1}^{m} \frac{\partial \zeta_k}{\partial c_n} \frac{\partial}{\partial \xi_k}. \tag{13}$$

The conditions for the differential operators in Eq. (9) allow an ambiguity in the overall normalizations. This ambiguity is fixed in Eq. (10) by setting the coefficients of the c_n-derivatives to be unity. This choice is very useful, since with this choice, the operators mutually commute: $[\nabla_n, \nabla_l] = 0$. This can be seen as follows. In general, $[\nabla_n, \nabla_l]$ is a linear differential operator. Since both ∇_n and ∇_l satisfy Eq. (9), their commutator $[\nabla_n, \nabla_l]$ should also satisfy Eq. (9). Then $[\nabla_n, \nabla_l]$ should again be given by a linear combination of $\{\nabla_n\}$:

$$[\nabla_n, \nabla_l] = \sum_k \alpha^{nlk} \nabla_k. \tag{14}$$

With the choice of Eq. (10), the left-hand side of Eq. (14) does not contain c_n-derivatives, while the right-hand side does. This means that $\alpha^{nlk} = 0$ and thus $[\nabla_n, \nabla_l] = 0$.

3. Physical implications for correlation numbers

The null identities in Eq. (6) provide the important fact that any correlation numbers of boundary-changing operators can be rewritten in terms of the correlation of boundary-preserving operators. The possibility is due to the fusion rule between BCO operators.

Let us consider the simplest case, $F_{11} = -\langle \log\det(R_{11}(Q))\rangle$, where $R_{11}(Q)$ is the 2×2 matrix

$$R_{11}(Q) = \begin{pmatrix} Q + a_1 & c \\ c & Q + a_2 \end{pmatrix}, \tag{15}$$

where the a_i are cosmological constants of (1,1) boundaries and assumed to take different values, $a_1 \neq a_2$. The off-diagonal component c couples to the boundary-changing operator B_{11} intertwining two different (1,1) boundaries and produces one null operator,

$$\nabla = \frac{\partial}{\partial c} + 2c \frac{\partial}{\partial \xi_2}, \tag{16}$$

with $\xi_1 = a_1 + a_2$ and $\xi_2 = a_1 a_2$. We have the null identity $\nabla^N F_{11} = 0$, where N is a positive integer. For $N = 1$, the identity shows

$$0 = \nabla F_{11}|_{c=0} = \left.\frac{\partial F_{11}}{\partial c}\right|_{c=0}, \tag{17}$$

which is consistent with the fact that the one-point correlation of BCOs is not allowed, since the boundary conditions contradict with each other. For $N = 2$, the two-point correlation of BCOs is given in terms of one-point boundary-preserving correlation numbers:

$$\langle B_{11} B_{11} \rangle = \left. \frac{\partial^2 F_{11}}{\partial c^2} \right|_{c=0} = -2\langle I_2 \rangle, \tag{18}$$

where we define

$$\langle {I_2}^k \rangle = \left. \frac{\partial^k F_{11}}{\partial \xi_2^k} \right|_{c=0}. \tag{19}$$

Using $\partial/\partial\xi_2 = -(1/a_{12})(\partial/\partial a_1 - \partial/\partial a_2)$ with $a_{12} = a_1 - a_2$, the result can be rewritten as

$$\langle I_2 \rangle = \frac{\langle O_1 \rangle - \langle O_2 \rangle}{a_{12}}, \tag{20}$$

where

$$\langle O_i \rangle = \left. \frac{\partial F_{11}}{\partial a_i} \right|_{c=0} = \left\langle \mathrm{tr} \frac{1}{Q + a_i} \right\rangle. \tag{21}$$

The one-point correlation $\langle O_i \rangle$ becomes $u^{1/b^2} \cosh\left(\frac{\pi s_i}{b}\right)$ if one evaluates it at the value $a_i = u \cosh(\pi b s_i)$, where u is a scale factor and s_i a boundary parameter.

It is noted that the free energy is given as

$$F_{11} = e^{-c^2 \frac{\partial}{\partial \xi_2}} F_{11}^{(D)}(a_1, a_2), \tag{22}$$

where $F_{11}^{(D)}$ is the c-independent part. This shows that the cubic correlation of the BCOs is absent, and the four-point correlation

$$\langle B_{11} B_{11} B_{11} B_{11} \rangle = \left. \frac{\partial^4 F_{11}}{\partial c^4} \right|_{c=0} = 12 \left. \frac{\partial^2 F_{11}}{\partial \xi_2^2} \right|_{c=0} = 12 \langle I_2 I_2 \rangle. \tag{23}$$

In a similar manner, from null identities obtained by successive applications of ∇, one can find identities relating higher-point correlation numbers of BCOs with lower-point correlation numbers of boundary-preserving operators.

One may look into a slightly complicated case: the BCO between the $(1, 1)$ boundary and $(1, 2)$ boundary. This can be investigated using $F_{12} = -\langle \log \det(R_{12}(Q)) \rangle$, where

$$R_{12}(Q) = \begin{pmatrix} Q + a_1 & c \\ c & (Q + a_2)(Q + a_3) \end{pmatrix}. \tag{24}$$

In this case also there is one off-diagonal parameter which couples to the BCO B_{12}. The null operator is given as

$$\nabla = \frac{\partial}{\partial c} + 2c \frac{\partial}{\partial \xi_3}, \tag{25}$$

with $\xi_3 = \mu_1\mu_2\mu_3$, and provides a similar null identity as between the (1, 1) boundaries: $\nabla^N F_{12} = 0$. It is obvious that one has an alternative expression of the free energy as in Eq. (22),

$$F_{11} = e^{-c^2 \frac{\partial}{\partial \xi_3}} F_{12}^{(D)}. \tag{26}$$

In this case as well, correlation numbers with insertions of an odd number of BCOs B_{12} are prohibited. Two-point correlation is similarly given as in Eq. (18),

$$\langle B_{12}B_{12}\rangle = -2 \left.\frac{\partial F_{12}}{\partial \xi_3}\right|_{c=0} = -2\langle I_3\rangle. \tag{27}$$

Here, $\langle I_3\rangle$ is given in terms of one-point correlations of the boundary-preserving operator O_i:

$$\langle B_{12}B_{12}\rangle = -2\left(\frac{\langle O_1\rangle}{a_{21}a_{31}} + \frac{\langle O_2\rangle}{a_{12}a_{32}} + \frac{\langle O_3\rangle}{a_{13}a_{23}}\right), \tag{28}$$

with $a_{ij} := a_i - a_j$. It is noted that the $(1, 2)$ boundary condition is realized when $a_2 = \mu_+$ and $a_3 = \mu_-$ with $\mu_\pm = u\cosh(\pi b(s_2 \pm ib))$ and s_2 real. In this case one has $\langle O_2\rangle|_{a_2=\mu_+} = \langle O_3\rangle|_{a_3=\mu_-} = -u^{1/b^2}\cosh(\pi b/s_2)$, and the two-point correlation of BCOs becomes[1]

$$\langle B_{12}B_{12}\rangle_* = -\frac{u^{\frac{1}{b^2}-2}\cosh\left(\frac{\pi s_p}{2b}\right)\cosh\left(\frac{\pi s_m}{2b}\right)}{\sinh\left(\frac{\pi b(s_p+ib)}{2}\right)\sinh\left(\frac{\pi b(s_p-ib)}{2}\right)\sinh\left(\frac{\pi b(s_m+ib)}{2}\right)\sinh\left(\frac{\pi b(s_m-ib)}{2}\right)}, \tag{29}$$

with $s_p = s_1 + s_2$ and $s_m = s_1 - s_2$ [9].

Suppose we consider a BCO between two different (1, 2) boundaries: $F_{22} = -\langle\log\det(R_{22}(Q))\rangle$, where

$$R_{22}(Q) = \begin{pmatrix} (Q + a_1)(Q + a_2) & c_1Q + c_2 \\ c_1Q + c_2 & (Q + a_3)(Q + a_4) \end{pmatrix}. \tag{30}$$

The off-diagonal terms have two real parameters c_1 and c_2, and thus there are two independent commuting null operators:

$$\nabla_1 = \frac{\partial}{\partial c_1} + 2\left(c_1\frac{\partial}{\partial \xi_2} + c_2\frac{\partial}{\partial \xi_3}\right), \tag{31}$$

$$\nabla_2 = \frac{\partial}{\partial c_2} + 2\left(c_1\frac{\partial}{\partial \xi_3} + c_2\frac{\partial}{\partial \xi_4}\right), \tag{32}$$

where ξ_i is defined by Eq. (12), implying null identities $\nabla_1^{N_1}\nabla_2^{N_2}F_{22} = 0$. The free energy can be written in the form

$$F_{22} = e^{-c_1^2\frac{\partial}{\partial \xi_2} - 2c_1c_2\frac{\partial}{\partial \xi_3} - c_2^2\frac{\partial}{\partial \xi_4}} F_{22}^{(D)}, \tag{33}$$

and therefore no correlations with odd numbers of BCOs $B_{22}^{(1)}$ and $B_{22}^{(2)}$, which are associated with the coupling constants c_1 and c_2, respectively, are allowed. There are three kinds of two-point

[1] The evaluation at specific values of boundary cosmological constants is indicated with a subscript asterisk hereafter.

correlations:

$$\langle B_{22}^{(1)} B_{22}^{(1)} \rangle = -2\langle I_2 \rangle = -2\left(\frac{a_1^2 \langle O_1 \rangle}{a_{21}a_{31}a_{41}} + \frac{a_2^2 \langle O_2 \rangle}{a_{12}a_{32}a_{42}} + \frac{a_3^2 \langle O_3 \rangle}{a_{13}a_{23}a_{43}} + \frac{a_4^2 \langle O_4 \rangle}{a_{14}a_{24}a_{34}} \right), \tag{34}$$

$$\langle B_{22}^{(1)} B_{22}^{(2)} \rangle = -2\langle I_3 \rangle = 2\left(\frac{a_1 \langle O_1 \rangle}{a_{21}a_{31}a_{41}} + \frac{a_2 \langle O_2 \rangle}{a_{12}a_{32}a_{42}} + \frac{a_3 \langle O_3 \rangle}{a_{13}a_{23}a_{43}} + \frac{a_4 \langle O_4 \rangle}{a_{14}a_{24}a_{34}} \right), \tag{35}$$

$$\langle B_{22}^{(2)} B_{22}^{(2)} \rangle = -2\langle I_4 \rangle = -2\left(\frac{\langle O_1 \rangle}{a_{21}a_{31}a_{41}} + \frac{\langle O_2 \rangle}{a_{12}a_{32}a_{42}} + \frac{\langle O_3 \rangle}{a_{13}a_{23}a_{43}} + \frac{\langle O_4 \rangle}{a_{14}a_{24}a_{34}} \right), \tag{36}$$

where $\langle B_{22}^{(i)} B_{22}^{(j)} \rangle = \partial^2 F_{22}/\partial c_i c_j|_{c=0}$ and $\langle I_i \rangle = \partial F_{22}^{(D)}/\partial \xi_i$.

To find BCO correlations between $(1,2)$ boundaries we need to use a correct parametrization of the a_i: $a_{1,2} = u \cosh(\pi b(s_1 \pm ib))$, $a_{3,4} = u \cosh(\pi b(s_2 \pm ib))$. It is notable that $\langle B_{22}^{(1)} B_{22}^{(2)} \rangle_* \neq 0$. One may find an orthogonal frame so that $\langle \widetilde{B}_{22}^{(1)} \widetilde{B}_{22}^{(2)} \rangle_* = 0$, where $\widetilde{B}_{22}^{(i)}$ is a new BCO associated with a new parameter q_i, defined by $c_1 Q + c_2 = q_1 P_1(Q) + q_2 P_0$ where $P_i(Q)$ is an ith-order polynomial in Q: $P_0 = 1$ and $P_1 = Q - \frac{\langle B_{22}^{(1)} B_{22}^{(2)} \rangle_*}{\langle B_{22}^{(2)} B_{22}^{(2)} \rangle_*}$, with

$$\frac{\langle B_{22}^{(1)} B_{22}^{(2)} \rangle_*}{\langle B_{22}^{(2)} B_{22}^{(2)} \rangle_*} = -\frac{u\,(\cosh(\pi b s_1) + \cosh(\pi b s_2))}{2\cos(\pi b^2)}, \tag{37}$$

as given in Ref. [13].

One can extend the discussion to BCOs between $(1, m_1)$ and $(1, m_2)$ boundaries without any difficulties using the null operator in Eq. (13). It is noted that $(\partial \zeta_k / \partial c_n)$ has no ξ_i dependence. As a result, the free energy has no correlations with odd-number insertions of BCOs.

4. Conclusion and discussion

In this paper we considered correlation numbers of boundary-changing and -preserving operators in the $(2, 2p+1)$ minimal Liouville gravity on disk. In terms of the one-matrix model, we showed that those correlation numbers satisfy some identities, called null identities in this paper. These identities enable us to express correlation numbers including boundary-changing operators in terms of correlation numbers with only boundary-preserving operators. This means that the correlation numbers of the boundary-changing operators can be determined from those of boundary-preserving operators. In addition, the null operator shows that the free energy has no cubic correlation of BCOs.

One may extend the matrix into $n \times n$ blocks to describe correlation numbers with n different boundary parameters:

$$F_{m_1 m_2 \cdots m_n} = -\langle \mathrm{tr} \log R_{m_1 m_2 \cdots m_n}(Q) \rangle,\ R_{m_1 m_2 \cdots m_n}(Q) = \begin{pmatrix} C_1(Q) & c_{12}(Q) & \dots & c_{1n}(Q) \\ c_{21}(Q) & C_2(Q) & \dots & c_{2n}(Q) \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}(Q) & c_{n2}(Q) & \dots & C_n(Q) \end{pmatrix}, \tag{38}$$

with $C_i(Q)$ and $c_{ij}(Q)$ respectively being

$$C_i(Q) = \prod_{k=1}^{m_i} \left(Q + a_k^{(i)} \right), \quad c_{ij}(Q) = c_{ji}(Q) = \sum_{n=0}^{d_{ij}} c_{d_{ij}+1-n}^{(ij)} Q^n, \tag{39}$$

where $d_{ij} = -1 + \min\left(m_i, m_j\right)$. The coefficients of $C_i(Q)$ and $c_{ij}(Q)$ are identified with the sources of boundary-preserving and -changing operators.

Under this setup, as opposed to the 2×2 block diagonal case, there seems in general no explicit formula for mutually commuting differential operators ∇_n that satisfies $\nabla_n\left(\det R_{m_1 m_2 \cdots m_n}(x)\right) = 0$. However, it is still possible to find them under the ansatz of Eq. (10) by requiring the conditions in Eq. (9), where the parameters $\{\zeta_i\}$ and $\{\xi_i\}$ are understood as straightforward extensions of Eqs. (8) and (12), respectively. For example, with a 3×3 block matrix,

$$F_{111} = -\left\langle \log\det(R_{111}(Q))\right\rangle, \quad R_{111}(Q) = \begin{pmatrix} Q + a_1 & c_3 & c_2 \\ c_3 & Q + a_2 & c_1 \\ c_2 & c_1 & Q + a_3 \end{pmatrix}. \tag{40}$$

There are three commuting differential operators that provide the null identities:

$$\nabla_i = \frac{\partial}{\partial c_i} + 2c_i \frac{\partial}{\partial \xi_2} + 2e_i \frac{\partial}{\partial \xi_3} \qquad (i = 1, 2, 3), \tag{41}$$

where $\xi_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$, $\xi_3 = a_1 a_2 a_3$. The coefficients e_i $(i = 1, 2, 3)$ are explicitly given by

$$\begin{aligned} e_1 &= \frac{a_2 a_{13}\, c_1 {c_2}^2 - a_3 a_{12}\, c_1 {c_3}^2 + a_1 a_{32}\, {c_1}^3 + a_{32} a_{12} a_{13}\,(c_3 c_2 - a_1 c_1)}{a_{32} {c_1}^2 - a_{31} {c_2}^2 + a_{21} {c_3}^2 - a_{21} a_{31} a_{32}}, \\ e_2 &= \frac{a_3 a_{21}\, c_2 {c_3}^2 - a_1 a_{23}\, c_2 {c_1}^2 + a_2 a_{13}\, {c_2}^3 + a_{13} a_{23} a_{21}\,(c_3 c_1 - a_2 c_2)}{a_{32} {c_1}^2 - a_{31} {c_2}^2 + a_{21} {c_3}^2 - a_{21} a_{31} a_{32}}, \\ e_3 &= \frac{a_1 a_{32}\, c_3 {c_1}^2 - a_2 a_{31}\, c_3 {c_2}^2 + a_3 a_{21}\, {c_3}^3 + a_{21} a_{31} a_{32}\,(c_2 c_1 - a_3 c_3)}{a_{32} {c_1}^2 - a_{31} {c_2}^2 + a_{21} {c_3}^2 - a_{21} a_{31} a_{32}}, \end{aligned} \tag{42}$$

which depend on the ξ_i, unlike in the 2×2 case. As a result, the free energy, satisfying the null identity $\nabla_1^{N_1} \nabla_2^{N_2} \nabla_3^{N_3} F_{111} = 0$, has non-vanishing cubic correlation of BCOs, for example

$$\left\langle B_{11}^{a_2 a_3} B_{11}^{a_3 a_1} B_{11}^{a_1 a_2} \right\rangle = \left.\frac{\partial^3 F_{111}}{\partial c_1 \partial c_2 \partial c_3}\right|_{c=0} = -2\left(\frac{a_{12}\left\langle O_3\right\rangle + a_{23}\left\langle O_1\right\rangle + a_{31}\left\langle O_2\right\rangle}{a_{12}\, a_{23}\, a_{31}}\right). \tag{43}$$

As one considers a larger matrix with block components of higher-order polynomials, their expression becomes more and more complicated, but one can still expect to find the differential operators case by case.

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