Le Grignou, Brice
Cubical model categories and quasi-categories. (English) Zbl 07205813
Theory Appl. Categ. 35, 622-664 (2020).

As was described in detail in [*D.-C. Cisinski*, Les préfaisceaux comme modèles des types d'homotopie. Paris: Société Mathématique de France (2006; Zbl 1111.18008)], upon which this paper hinges greatly, there exists a category of cubes $\square_p$ (the subscript "p" stands for "pre"), which roughly consists of cubes of various dimensions $\square_p^n$, together with face inclusions $\delta_i : \square_p^n \to \square_p^{n+1}$ and contractions $\sigma_i : \square_p^n \to \square_p^{n-1}$. The category has a monoidal product

$$\square_p^n \otimes \square_p^m = \square_p^{n+m}$$

We have the category of precubical sets

$$\square_p - \mathrm{Set} = \mathrm{Fun}\left(\square_p^{\mathrm{op}}, \mathrm{Set}\right)$$

just as we have the category of simplicial sets

$$\mathrm{sSet} = \mathrm{Fun}\left(\triangle^{\mathrm{op}}, \mathrm{Set}\right)$$

It was established there that

Proposition. Let $(\boldsymbol{E}, \otimes, \mathbf{1})$ be a monoidal model category. Then the data of an interval H of $\boldsymbol{E}$ is essentially the data of a monoidal Quillen adjunction

$$\square_p - \mathrm{Set} \underset{R_p^H}{\overset{L_p^H}{\rightleftarrows}} \boldsymbol{E}$$

inducing another adjunction

$$\mathrm{Cat}_{\square_p} \underset{R_p^H}{\overset{L_p^H}{\rightleftarrows}} \mathrm{Cat}_{\boldsymbol{E}}$$

between the category enriched in $\square_p$ and that enriched in $\boldsymbol{E}$, which is a Quillen adjunction when the Dwyer-Kan model structure on $\mathrm{Cat}_{\boldsymbol{E}}$ exists.

The principal objective in this paper is to make use of cubical sets systematically to investigate enriched categories. The main result is as follows:

Theorem. Let $\boldsymbol{E}$ be a monoidal model category and suppose that the category $\mathrm{Cat}_{\boldsymbol{E}}$ of categories enriched over $\boldsymbol{E}$ has a Dwyer-Kan model structure. Then, for any Reedy cofibrant replacement F of the cosimplicial $\boldsymbol{E}$-enriched category $n \mapsto [n]$, the induced adjunctione

$$\mathrm{sSet} \underset{F^!}{\overset{F_!}{\rightleftarrows}} \mathrm{Cat}_{\boldsymbol{E}}$$

is a Quillen adjunction.

The paper consists of five sections. §1 describes in detail the theory of enriched categories together with some results about their homotopy theory. §2 and§3 are concerned with the categories of precubical sets and cubical sets with connections as well as their model structures. §4 addresses links between quasi-categories and categories enriched in cubical sets, which, given a monoidal category $\boldsymbol{E}$ equipped with an interval, allows of precise conditions relating the Joyal model category of simplicial sets to the model category $\mathrm{Cat}_{\boldsymbol{E}}$ a Quillen adjunction. The goal of §5 is to describe various contexts where cubical categories appear.

The idea that the homotopy coherent nerve functors for dg categories and simplicial categories factor through categories enriched over cubical sets with connections was already discussed in [*M. Rivera* and

M. Zeinalian, Algebr. Geom. Topol. 18, No. 7, 3789–3820 (2018; Zbl 1423.18026)]. Similar ideas to those in §4 already appeared independently in [https://www.uwo.ca/math/faculty/kapulkin/papers/cubical-approach-to-straightening.pdf].

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18D20 Enriched categories (over closed or monoidal categories)
18N50 Simplicial sets, simplicial objects

Keywords:

enriched categories; cubical sets

Full Text: Link

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