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Weak model categories in classical and constructive mathematics. (English) Zbl 07216042
Theory Appl. Categ. 35, 875-958 (2020).

This paper is concerned with a weakening of *D. G. Quillen* [Homotopical algebra. Berlin-Heidelberg-New York: Springer-Verlag (1967; Zbl 0168.20903)] *model categories* other than *left semi-model categories* [<https://web.math.rochester.edu/people/faculty/doug/otherpapers/spitzweck.pdf>] and *right semi-model categories* [*C. Barwick*, Homology Homotopy Appl. 12, No. 2, 245–320 (2010; Zbl 1243.18025)]. The paper introduces a new notion of *weak model category*, in which it is only required that arrows with both a cofibrant domain and a fibrant target can be factored, and the lifting property between (acyclic) cofibrations with fibrant domains and (acyclic) fibrations with fibrant targets is asked for. The notion is self-dual, being a generalization of the familiar notion of Quillen model category which encompasses both left and right semi-model categories. The author defines the homotopy category of a weak model category, a notion of Quillen adjunction and Quillen equivalence between weak model categories, and so on as in the theory of Quillen model categories. The guiding principle of the author is that only the notion of cofibration with cofibrant domain and that of fibration with fibrant target should be considered meaningful.

It is shown that the projective model structure on chain complexes, the Kan-Quillen model structure on simplicial set and the Verity model structure on stratified simplicial sets can all be proved to exist constructively as weak model structures, opening the door to a constructive theory of higher categories, which has, in [*N. Gambino* and *S. Henry*, “Towards a constructive simplicial model of Univalent Foundations”, Preprint, [arXiv:1905.06281](https://arxiv.org/abs/1905.06281)], allowed, up to some coherent issue that still is to be taken care of, to give a constructive version of Voevodsky’s simplicial model of homotopy type theory. It was shown in [*N. Gambino* et al., “The constructive Kan-Quillen model structure: two new proofs”, Preprint, [arXiv:1907.05394](https://arxiv.org/abs/1907.05394), *S. Henry*, “A constructive account of the Kan-Quillen model structure and of Kan’s Ex^∞ functor”, Preprint, [arXiv:1905.06160](https://arxiv.org/abs/1905.06160)] that the Kan-Quillen model structure on simplicial sets being a proper Quillen model category is to be constructively proved, though the author admits frankly that a similar result for the Joyal model structure is out of reach at the moment.

This paper concentrates on the aspect of the theory of weak model categories that can be developed within constructive mathematics. This means that many classical topics of the theory of Quillen model categories such as the notion of combinatorial model structure and the theory of Bousfield localizations are not even touched upon. The author addresses these non-constructive aspects as well as the theory of combinatorial and accessible weak model categories and the precise relation weak model structures and left and right semi-model structures in [*S. Henry*, “Combinatorial and accessible weak model categories”, Preprint, [arXiv:2005.02360](https://arxiv.org/abs/2005.02360)].

Another advantage of weak model categories over Quillen model categories is several easy criteria for constructing a weak model structure on a category, particularly in the case where we start from the cofibrations and the fibrations without having a good description of the weak equivalences, as this is in general a challenging task for Quillen model categories. This is a key point for constructing constructively examples of weak model structures.

Reviewer: [Hirokazu Nishimura \(Tsukuba\)](#)

MSC:

- [55U35](#) Abstract and axiomatic homotopy theory in algebraic topology
- [55U40](#) Topological categories, foundations of homotopy theory
- [18N40](#) Homotopical algebra, Quillen model categories, derivators
- [18N50](#) Simplicial sets, simplicial objects
- [18N65](#) (∞, n) -categories and (∞, ∞) -categories

Keywords:

model categories; constructive mathematics; simplicial sets; semi-simplicial sets; complicial sets

Full Text: [Link](#)

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