**Ara, Dimitri; Lucas, Maxime**

**The folk model category structure on strict $\omega$-categories is monoidal.** (English) Zbl 07216039

Theory Appl. Categ. 35, 745-808 (2020).

The category $\omega - \mathcal{C}at$ of strict $\omega$-categories is endowed with the *folk model category structure* [*Y. Lafont* et al., Adv. Math. 224, No. 3, 1183–1231 (2010; Zbl 1236.18017)]. On the other hand, the category $\omega - \mathcal{C}at$ is endowed with two non-trivial monoidal category structure, namely, the *Gray tensor product* and the *join*. The first monoidal category structure was introduced in [*F. A. Al-Agl* and *R. Steiner*, Proc. Lond. Math. Soc. (3) 66, No. 1, 92–128 (1993; Zbl 0742.18010)], and then studied in [`libcat.calacademy.org/title/on-combinatorial-models-for-higher-dimensional-homotopies`], This tensor product generalizes the tensor product of 2-categories introduced by *J. W. Gray* [Formal category theory: Adjointness for 2-categories. Berlin-Heidelberg-New York: Springer-Verlag (1974; Zbl 0285.18006)]. The second monoidal category structure is given by the join of $\omega$-categories $*$, introduced by the first author of this paper and *G. Maltsiniotis* in ["Joint et tranches pour les $\infty$-catégories strictes", Mém. Soc. Math. Fr. 165 (2020)] to study slice $\omega$-categories in a similar way as Joyal did for quasi-categories.

The principal objective in this paper is to show that both the Gray tensor product and the join interact well with the folk model category structure. The paper consists of 7 sections with an appendix on monoidal model categories and derived tensor products. §1 recalls the definitions related to the folk model category structure on the category $\omega - \mathcal{C}at$ of strict $\omega$-categories. The aim of §2 is to recall the definition of the Gray tensor product. §3 aims to establish the pushout-product, for the Gray tensor product, of two folk cofibrations is a folk cofibration. §4 establishes that if $X$ is a folk cofibrant $\omega$-category, then $J_1 \otimes X$ with $J_1$ being the $\omega$-category obtained by the factorization of the codiagonal of the terminal $\omega$-category into a cofibration followed by a trivial fibration is a cylinder object for $X$ in the folk model category. §5 makes of $\omega - \mathcal{C}at$ with the folk model category structure a monoidal model category. §6 introduces the category $(m, n) - \mathcal{C}at$ of strict $(m, n)$-categories, investigating the interactions between the Gray tensor product and these $(m, n)$-categories. §7 recalls the definition of the join of $\omega$-categories. It is shown, as in §5, that the join makes of $\omega - \mathcal{C}at$ with the folk model category structure a monoidal model category.

Reviewer: Hirokazu Nishimura (Tsukuba)

**MSC:**

18M05 Monoidal categories, symmetric monoidal categories
18N30 Strict omega-categories, computads, polygraphs
18N40 Homotopical algebra, Quillen model categories, derivators
55U35 Abstract and axiomatic homotopy theory in algebraic topology

**Keywords:**

augmented directed complexes; folk model category structure; Gray tensor product; join; locally biclosed monoidal categories; monoidal model categories; oplax transformations; strict $\omega$-categories; strict $\omega$-groupoids; strict $(m, n)$-categories

**Full Text:** Link

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