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Atiyah classes of strongly homotopy Lie pairs. (English) Zbl 07062267
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Strongly homotopy (SH) *Lie algebras* were introduced in [*T. Lada* and *M. Markl*, Commun. Algebra 23, No. 6, 2147–2161 (1995; Zbl 0999.17019); *T. Lada* and *J. Stasheff*, Int. J. Theor. Phys. 32, No. 7, 1087–1103 (1993; Zbl 0824.17024)]. They were addressed mainly in mathematical physics and supergeometry. The paper [*D. Bashkirov* and *A. A. Voronov*, J. Homotopy Relat. Struct. 12, No. 2, 305–327 (2017; Zbl 1371.18015)] used the Batalin-Vilkovisky formalism to regard an SH Lie algebra as a special pointed BV_∞-manifold.

The *Atiyah class* of a holomorphic vector bundle U over a complex manifold was introduced in [*M. F. Atiyah*, Trans. Am. Math. Soc. 85, 181–207 (1957; Zbl 0078.16002)] as the obstruction to the existence of a holomorphic connection on U. *P. Molino* [C. R. Acad. Sci., Paris, Sér. A 272, 779–781 (1971; Zbl 0211.26701); Topology 12, 317–325 (1973; Zbl 0269.57014)] defined the Atiyah-Molino class of a foliation of a manifold for capturing the existence of a locally projectable connection. The Atiyah class of a LIe algebra pair was investigated in [*H.-C. Wang*, Nagoya Math. J. 13, 1–19 (1958; Zbl 0086.36502); *V. H. Nguyen*, C. R. Acad. Sci., Paris 260, 45–48 (1965; Zbl 0144.21101); *M. Bordemann*, "Atiyah classes and equivariant connections on homogeneousspaces", Travaux Math. 20, 29–82 (2012)] to characterize the existence of invariant connections on a homogeneous space. Kontsevich's seminal work on deformation quantization [*M. Kontsevich*, Lett. Math. Phys. 48, No. 1, 35–72 (1999; Zbl 0945.18008); Lett. Math. Phys. 66, No. 3, 157–216 (2003; Zbl 1058.53065)] has inspired renewed interest in Atiyah classes, which are also related to the Rozansky-Witten theory [*M. Kapranov*, Compos. Math. 115, No. 1, 71–113 (1999; Zbl 0993.53026); *L. Rozansky* and *E. Witten*, Sel. Math., New Ser. 3, No. 3, 401–458 (1997; Zbl 0908.53027)]. *D. Calaque* and *M. Van den Bergh* considered the Atiyah class of a DG module over a DG algebra in [Adv. Math. 224, No. 5, 1839–1889 (2010; Zbl 1197.14017)], claiming that, given a Lie algebra pair $(\mathfrak{d},\mathfrak{g})$, the Atiyah class of the quotient $\mathfrak{d}/\mathfrak{g}$ coincides with the class for the obstruction to the PBW problem studied in [*D. Calaque* et al., J. Algebra 378, 64–79 (2013; Zbl 1316.17008); *D. Calaque*, Rend. Semin. Mat. Univ. Padova 131, 23–47 (2014; Zbl 1337.53098); *D. Grinberg*, Poincaré-Birkhoff-Witt type results for inclusions of Lie algebras. Massachusetts Institute of Technology (Master Thesis) (2011)].

This paper studies SH Lie algebras to show that Atiyah classes exist and play the role of sending homotopical objects to Lie objects. It is shown that, for any SH Lie pair (L,A) and an A-module E, one can extend the A-module structure on E to an L-connection ∇ on E. The curvature R^∇ measures the failure of E being an L-module. From R^∇, the authors extract a particular element

$$\alpha^E_\nabla := (J\otimes 1)\left(R^\nabla\right) \in \mathcal{O}(A)\otimes A^\perp\otimes \mathrm{End}(E)$$

which is a cocycle of degree 2 and is called the Atiyah cocyle of the SH Lie pair (L,A) with respect to the A-module E and the L-connection ∇, where $\mathcal{O}(A)$ is the graded algebra of formal power series on A. The paper contains three main results. The first main result (Theorem 3.4) of the paper is

Theorem. The cohomology class

$$\left[\alpha^E_\nabla\right] \in H^2_{\mathrm{CE}}\left(A, A^\perp\otimes \mathrm{End}(E)\right)$$

called the Atiyah class is independent of the choice of ∇. In particular, with the canonical A-module L/A, we have the canonical Atiyah class

$$\left[\alpha^{L/A}\right] \in H^2_{\mathrm{CE}}\left(A, \mathrm{Hom}\left(L/A\otimes L/A, L/A\right)\right)$$

§4 introduces the Atiyah operator and Atiyah functor. The second main result (Theorem 4.4) of the paper, which is a generalization of the corresponding results in [*F. Bottacin*, in: Current trends in analysis and its applications. Proceedings of the 9th ISAAC congress, Kraków, Poland, August 5–9, 2013. Cham: Birkhäuser/Springer. 375–393 (2015; Zbl 1325.32029)]; *Z. Chen* et al., Commun. Math. Phys. 341, No. 1, 309–349 (2016; Zbl 1395.53090); *M. Kapranov*, Compos. Math. 115, No. 1, 71–113 (1999; Zbl 0993.53026)], goes as follows:

Theorem. The graded vector space $H^{\cdot}_{\mathrm{CE}}(A,(L/A)[-2])$ with the binary operation induced by the Atiyah operator $\alpha^{L/A}$ is a Lie algebra. Furthermore, if E is an A-module, then $H^{\cdot}_{\mathrm{CE}}(A,E)$ is a Lie algebra module over $H^{\cdot}_{\mathrm{CE}}(A,(L/A)[-2])$, with the action induced by the Atiyah operator α^{E}.

§5 studies a special kind of deformation of the given SH Lie pair (L,A), namely, A-compatible infinitesimal deformations of L. The third main result (Theorem 5.3) of the paper goes as follows:

Theorem. If two A-compatible infinitesimal deformations of L are gauge equivalent, then the two associated Atiyah classes coincide.

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MSC:

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16E45 Differential graded algebras and applications (associative algebraic aspects)
58C50 Analysis on supermanifolds or graded manifolds

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homotopical algebra; L_∞-algebra; Atiyah class

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