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Higher orbifolds and Deligne-Mumford stacks as structured infinity-topoi. (English)

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Deligne-Mumford stacks, which are locally modeled by quotients of schemes by finite group actions, and are used to model moduli spaces of interesting automorphisms such as elliptic curves, were introduced in [*P. Deligne* and *D. Mumford*, Publ. Math., Inst. Hautes Étud. Sci. 36, 75–109 (1969; Zbl 0181.48803); Matematika, Moskva 16, No. 3, 13–53 (1972; Zbl 0233.14008)]. The theory of Deligne-Mumford stacks has lead to splendid developments in algebraic geometry [*L. Lafforgue*, J. Am. Math. Soc. 11, No. 4, 1001–1036 (1998; Zbl 1045.11041)] and Gromov-Witten theory [*M. Kontsevich*, Commun. Math. Phys. 147, No. 1, 1–23 (1992; Zbl 0756.35081)]. Higher categorical generalizations of Deligne-Mumford stacks have been attempted in [*B. Toën* and *G. Vezzosi*, Homotopical algebraic geometry. II: Geometric stacks and applications. Providence, RI: American Mathematical Society (AMS) (2008; Zbl 1145.14003); *J. Lurie*, "Derived algebraic geometry. V: Structured spaces", Preprint, `arXiv:0905.0459`] to find out far reaching generalizations of algebraic geometry such as the algebraic geometry of simplicial commutative rings and the algebraic geometry of $\mathbb{E}_\infty$-ring spectra.

The notion of a *smooth orbifold* grew out of differential topology and the study of automorphic forms, also appearing in the study of 3-manifolds and in foliation theory. Although orbifolds were initially defined as topological spaces with an orbifold atlas, they are now encoded as differentiable stacks [*D. A. Pronk*, Compos. Math. 102, No. 3, 243–303 (1996; Zbl 0871.18003)]. *Étale differentiable stacks* are differentiable stacks which are mild generalizations of smooth orbifolds, subsuming leaf spaces of foliated manifolds and quotients of manifolds by almost-free Lie group actions [*D. Carchedi*, Algebr. Geom. Topol. 13, No. 2, 831–903 (2013; Zbl 1263.22001); Adv. Math. 352, 56–132 (2019; Zbl 07082639); *R. Hepworth*, Algebr. Geom. Topol. 9, No. 2, 1105–1175 (2009; Zbl 1175.55004); Theory Appl. Categ. 22, 542–587 (2009; Zbl 1206.37009); *I. Moerdijk*, Rev. Acad. Ci. Exactas Fís. Quím. Nat. Zaragoza, II. Ser. 48, 5–33 (1993; Zbl 0804.57014); *D. A. Pronk*, Compos. Math. 102, No. 3, 243–303 (1996; Zbl 0871.18003); *G. Trentinaglia* and *C. Zhu*, Compos. Math. 148, No. 3, 807–834 (2012; Zbl 1245.18005); *H.-H. Tseng* and *C. Zhu*, Compos. Math. 142, No. 1, 251–270 (2006; Zbl 1111.58019); *C. Wockel*, Adv. Math. 228, No. 4, 2218–2257 (2011; Zbl 1238.22012)].

This monograph develops a universal framework to study higher étale differentiable stacks and orbifolds on the one hand, and higher algebraic Deligne-Mumford stacks as well as their derived and spectral analogues on the other. This framework was developed not only to provide a useful language putting orbifolds and Deligne-Mumford stacks on the same footing but also to reveal new insights about these objects and their generalizations. In particular, a new characterization of classical Deligne-Mumford stacks as well as its extension to the derived and spectral setting can be obtained, while, in the differentiable setting, we find out that there is a natural correspondence between n-dimensional higher étale differentiable stacks and classical fields for n-dimensional field theories in the sense of [*D. S. Freed* and *C. Teleman*, Commun. Math. Phys. 326, No. 2, 459–476 (2014; Zbl 1285.81057)].

An ∞-*topos* is an $(\infty,1)$-category of ∞-sheaves, ∞-sheaves being higher categorical versions of sheaves and stacks taking values in in the $(\infty,1)$-category $\boldsymbol{Grp}_\infty$ of ∞-groupoids. An ∞*groupoid* is a model for the homotopy type of a topological space, so that ∞-sheaves are essentially homotopy sheaves of spaces. The homeomorphism type of a topological space X is to be recovered from its topos $Sh(X)$ of sheaves of sets , and a general topos is to be put down as a generalized space with points being of automorphisms. The homeomorphism type of X is also encoded by the ∞-topos $Sh_\infty(X)$ of ∞-sheaves of ∞-groupoids over X, and a general ∞-topos is to be put down as a generalized space with points possessing automorphisms, automorphisms between automorphisms, automorphisms between automorphisms between automorphisms, and so on. The idea is the same geometric intuition behind the underlying space of a higher orbifold or higher Deligne-Mumford stack. The author models all the geometric objects as ∞-topoi with an appropriate structure sheaf. The author starts by specifying *local models*, generalized orbifolds or Deligne-Mumford stacks being defined as those objects locally equivalent to the local models, in much the same way as one builds manifolds out of Euclidean spaces or schemes our of affine schemes.

The monograph consists of 6 chapters. Chapter 1 is an introduction and an overview. Chapter 2 offers a brief review of higher topos theory. develops the role of n-topoi ($0 \leq n \leq \infty$) as generalized spaces, paying particular attention to the notion of a local homeomorphism between n-topoi and introducing a Grothendieck topology (called the étale topology) on higher topoi, by which one can glue higher topoi together along local homeomorphisms. Chapter 4 introduces the concept of structured ∞-topoi and reviews the notions of geometry and geometric structure in [*J. Lurie*, "Derived algebraic geometry. V: Structured spaces", Preprint, arXiv:0905.0459].

Chapter 5 is the cornerstone of the monograph, developing a natural framework to model generalized higher orbifolds and Deligne-Mumford stacks as structured ∞-topoi. The following four theorems are established.

Theorem. There is a full and faithful functor

$$\overline{\mathcal{L}} \hookrightarrow Sh_{\infty}(\mathcal{L})$$

between the $(\infty, 1)$-category of $\mathcal{L}$-étendues and the $(\infty, 1)$-category of ∞-sheaves over $\mathcal{L}$.

Theorem. If the $(\infty, 1)$-category of local models $\mathcal{L}$ is essentially small, then there is a canonical equivalence of $(\infty, 1)$-categories

$$\overline{\mathcal{L}}^{\acute{e}t} \simeq Sh_{\infty}\left(\mathcal{L}^{\acute{e}t}\right)$$

between the $(\infty, 1)$-category of $\mathcal{L}$-étendues and only their étale morphisms, and the $(\infty, 1)$-category of ∞-sheaves over the $(\infty, 1)$-category consisting of the objects of $\mathcal{L}$-étendues and only their étale morphisms. Moreover, the ∞-topos $U := Sh_{\infty}\left(\mathcal{L}^{\acute{e}t}\right)$ carries a canonical structure sheaf $\mathcal{O}_U$, making the pair $(U, \mathcal{O}_U)$ an $\mathcal{L}$-étendue. Furthermore, $(U, \mathcal{O}_U)$ is a terminal object in $\overline{\mathcal{L}}^{\acute{e}t}$.

Theorem. An ∞-sheaf $\mathcal{X}$ on $\overline{\mathcal{L}}$ is Deligne-Mumford iff it is in the essential image of the étale prolongation functor

$$j_! : Sh_{\infty}\left(\mathcal{L}^{\acute{e}t}\right) \to Sh_{\infty}(\mathcal{L})$$

Theorem. An ∞-sheaf $\mathcal{X}$ on $\overline{\mathcal{L}}$ is Deligne-Mumford iff it is in the essential image of a relative prolongation functor

$$j_!^{\mathcal{U}} : Sh_{\infty}\left(\mathcal{L}^{\acute{e}t}\right) \to Sh_{\infty}(\mathcal{L})$$

for some small subcategory $\mathcal{U}$ of $\mathcal{L}$.

Chapter 6 spells out the general machinery developed in Chapter 5 within the setting of differential topology, and in the cases of classical, derived and spectral algebraic geometry. §6.1 defines the concept of a differentiable ∞-orbifold and an étale differentiable ∞-stack. The main results are the following two theorems.

Theorem. An ∞-sheaf $\mathcal{X}$ on the category of smooth manifolds Mfd is an étale differentiable stack iff it is étale differentiable ∞-stack and is 1-truncated. Similarly, $\mathcal{X}$ is a differentiable orbifold iff it is a differentiable ∞-orbifold and is 1-truncated.

Theorem. Any étale differentiable ∞-stack $\mathcal{X}$ can be realized as the étale space of an $\mathcal{X}$-gerbe over an effective étale differentiable ∞-stack $\mathcal{Y}$.

§6.2 applies the results of Chapter 5 to the setting of $\mathcal{G}$-schemes for a geometry in the sense of [*J. Lurie*, "Derived algebraic geometry. V: Structured spaces", Preprint, arXiv:0905.0459], working out consequences of these results to the theory of higher Deligne-Mumford stacks, derived Deligne-Mumford stacks and spectral Deligne-Mumford stacks. The main results are the following theorem and a list of theorems as special cases of the second, third and fourth theorems above.

Theorem. Let k be a commutative ring and let $\mathcal{L}$ be the category of affine k-schemes, viewed as a category of structured ∞-topoi, with each commutative k-algebra A being associated to its small étale ∞-topos $Sh_{\infty}(A_{\acute{e}t})$. An ∞-sheaf on affine k-schemes with respect to the étale topology is a Deligne-Mumford stack in the classical sense iff it is a Deligne-Mumford stack for $\mathcal{L}$ and is 1-truncated.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18-02 Research exposition (monographs, survey articles) pertaining to category theory
18B25 Topoi
14A30 Fundamental constructions in algebraic geometry involving higher and derived categories (homotopical algebraic geometry, derived algebraic geometry, etc.)
18N60 $(\infty, 1)$-categories (quasi-categories, Segal spaces, etc.); ∞-topoi, stable ∞-categories
14D23 Stacks and moduli problems
58A03 Topos-theoretic approach to differentiable manifolds

Keywords:

infinity-topoi; higher stacks; Deligne-Mumford stacks; orbifolds

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References:

[1] Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos, Lecture Notes in Mathematics, Vol. 269, xix+525 pp. (1972), Springer-Verlag, Berlin-New York

[2] Abramovich, Dan; Vistoli, Angelo, Compactifying the space of stable maps, J. Amer. Math. Soc., 15, 1, 27-75 (2002) · Zbl 0991.14007

[3] Baez, John C.; Shulman, Michael, Lectures on \(n\)-categories and cohomology. Towards higher categories, IMA Vol. Math. Appl. 152, 1-68 (2010), Springer, New York · Zbl 1191.18008

[4] Birkhoff, Garrett, Von Neumann and lattice theory, Bull. Amer. Math. Soc., 64, 50-56 (1958) · Zbl 0080.00412

[5] Breen, Lawrence, On the classification of \(2\)-gerbes and \(2\)-stacks, Astérisque, 225, 160 pp. (1994) · Zbl 0818.18005

[6] Calaque, Damien; Pantev, Tony; Toën, Bertrand; Vaquié, Michel; Vezzosi, Gabriele, Shifted Poisson structures and deformation quantization, J. Topol., 10, 2, 483-584 (2017) · Zbl 1428.14006

[7] Carchedi, David, An étalé space construction for stacks, Algebr. Geom. Topol., 13, 2, 831-903 (2013) · Zbl 1263.22001

[8] Carchedi, David, Étale stacks as prolongations, Adv. Math., 352, 56-132 (2019) · Zbl 07082639

[9] Deligne, P.; Mumford, D., The irreducibility of the space of curves of given genus, Inst. Hautes Études Sci. Publ. Math., 36, 75-109 (1969) · Zbl 0181.48803

[10] Dubuc, Eduardo J., Sur les modèles de la géométrie différentielle synthétique, Cahiers Topologie Géom. Différentielle, 20, 3, 231-279 (1979) · Zbl 0473.18008

[11] Freed, Daniel S.; Teleman, Constantin, Relative quantum field theory, Comm. Math. Phys., 326, 2, 459-476 (2014) · Zbl 1285.81057

[12] Giraud, Jean, Cohomologie non abélienne, ix+467 pp. (1971), Springer-Verlag, Berlin-New York · Zbl 0226.14011

[13] Alexander Grothendieck. Recoltes et Semailles. Universite des Sciences et Techniques du Languedoc, Montpellier, 1985-1987.

[14] Haefliger, André, Homotopy and integrability. Manifolds-Amsterdam 1970 (Proc. Nuffic Summer School), Lecture Notes in Mathematics, Vol. 197, 133-163 (1971), Springer, Berlin

[15] Hepworth, Richard, Morse inequalities for orbifold cohomology, Algebr. Geom. Topol., 9, 2, 1105-1175 (2009) · Zbl 1175.55004

[16] Hepworth, Richard, Vector fields and flows on differentiable stacks, Theory Appl. Categ., 22, 542-587 (2009) · Zbl 1206.37009

[17] Isbell, John R., Atomless parts of spaces, Math. Scand., 31, 5-32 (1972) · Zbl 0246.54028

[18] Johnstone, Peter T., Sketches of an elephant: a topos theory compendium. Vol. 2, Oxford Logic Guides 44, i-xxii, 469-1089 and I1-I71 (2002), The Clarendon Press, Oxford University Press, Oxford · Zbl 1071.18002

[19] Joyal, A., Quasi-categories and Kan complexes, J. Pure Appl. Algebra, 175, 1-3, 207-222 (2002) · Zbl 1015.18008

[20] Joyal, André; Tierney, Myles, An extension of the Galois theory of Grothendieck, Mem. Amer. Math. Soc., 51, 309, vii+71 pp. (1984) · Zbl 0541.18002

[21] Kontsevich, Maxim, Intersection theory on the moduli space of curves and the matrix Airy function, Comm. Math. Phys., 147, 1, 1-23 (1992) · Zbl 0756.35081

[22] Lafforgue, Laurent, Une compactification des champs classifiant les chtoucas de Drinfeld, J. Amer. Math. Soc., 11, 4, 1001-1036 (1998) · Zbl 1045.11041

[23] Jacob Lurie. Derived algebraic geometry V: Structured spaces. \href{http://arxiv.org/abs/0905.0459}arXiv:0905.0459, 2009.

[24] Lurie, Jacob, Higher topos theory, Annals of Mathematics Studies 170, xviii+925 pp. (2009), Princeton University Press, Princeton, NJ · Zbl 1175.18001

[25] Jacob Lurie. Derived algebraic geometry VII: Spectral schemes, 2011. \href{http://www.math.harvard.edu/ lurie/papers/DAG-

VII.pdf}http://www.math.harvard.edu/ lurie/papers/DAG-VII.pdf.

[26] Mac Lane, Saunders; Moerdijk, Ieke, Sheaves in geometry and logic, Universitext, xii+629 pp. (1994), Springer-Verlag, New York · Zbl 0822.18001

[27] McLarty, Colin, The uses and abuses of the history of topos theory, British J. Philos. Sci., 41, 3, 351-375 (1990) · Zbl 0709.18002

[28] Moerdijk, I.; Pronk, D. A., Orbifolds, sheaves and groupoids, \(K\)-Theory, 12, 1, 3-21 (1997) · Zbl 0883.22005

[29] Moerdijk, Ieke, The classifying topos of a continuous groupoid. I, Trans. Amer. Math. Soc., 310, 2, 629-668 (1988) · Zbl 0706.18007

[30] Moerdijk, Ieke, Foliations, groupoids and Grothendieck étendues, Rev. Acad. Cienc. Zaragoza (2), 48, 5-33 (1993) · Zbl 0804.57014

[31] Pantev, Tony; Toën, Bertrand; Vaquié, Michel; Vezzosi, Gabriele, Shifted symplectic structures, Publ. Math. Inst. Hautes Études Sci., 117, 271-328 (2013) · Zbl 1328.14027

[32] Pronk, Dorette A., Etendues and stacks as bicategories of fractions, Compositio Math., 102, 3, 243-303 (1996) · Zbl 0871.18003

[33] Reyes, G. E.; Wraith, G. C., A note on tangent bundles in a category with a ring object, Math. Scand., 42, 1, 53-63 (1978) · Zbl 0392.18011

[34] Satake, I., On a generalization of the notion of manifold, Proc. Nat. Acad. Sci. U.S.A., 42, 359-363 (1956) · Zbl 0074.18103

[35] Spivak, David I., Derived smooth manifolds, Duke Math. J., 153, 1, 55-128 (2010) · Zbl 1420.57073

[36] Toën, Bertrand; Vezzosi, Gabriele, Homotopical algebraic geometry. II. Geometric stacks and applications, Mem. Amer. Math. Soc., 193, 902, x+224 pp. (2008) · Zbl 1145.14003

[37] Trentinaglia, Giorgio; Zhu, Chenchang, Strictification of étale stacky Lie groups, Compos. Math., 148, 3, 807-834 (2012) · Zbl 1245.18005

[38] Tseng, Hsian-Hua; Zhu, Chenchang, Integrating Lie algebroids via stacks, Compos. Math., 142, 1, 251-270 (2006) · Zbl 1111.58019

[39] Wockel, Christoph, Categorified central extensions, étale Lie 2-groups and Lie's third theorem for locally exponential Lie algebras, Adv. Math., 228, 4, 2218-2257 (2011) · Zbl 1238.22012

[40] Wolfson, Jesse, Descent for \(n\)-bundles, Adv. Math., 288, 527-575 (2016) · Zbl 1330.18021

[41] Zhu, Chenchang, \(n\)-groupoids and stacky groupoids, Int. Math. Res. Not. IMRN, 21, 4087-4141 (2009) · Zbl 1180.22006