Baez, John C.; Fong, Brendan

A Noether theorem for Markov processes. (English) Zbl 1280.81015

J. Math. Phys. 54, No. 1, 013301, 8 p. (2013).

There is a rich analogy between quantum mechanics and so-called stochastic mechanics, where probabilities take place of amplitudes [*J. C. Baez* and *J. D. Biamonte*, Quantum techniques in stochastic mechanics. Hackensack, NJ: World Scientific (2018; Zbl 1405.81005)]. In quantum mechanics, the state of a system is specified by an element ψ of a Hilbert space, and its time evolution is described by the Schrödinger equation

$$\frac{d}{dt}\psi = -iH\psi$$

where H is a self-adjoint linear operator called the Hamiltonian. For Markov processes, the state of a system is specified by a probability distribution ψ on some measure space, and its time evolution is described by the so-called master equation

$$\frac{d}{dt}\psi = H\psi$$

where H is linear operator known as a stochastic Hamiltonian, transition rate matrix or intensity matrix.

It is well-known in quantum mechanics that conserved quantities correspond to self-adjoint operators commuting with the Hamiltonian. The principal objective in this paper is to present a similar result for Markov processes. In the Hamiltonian approach to classical mechanics, an observable having vanishing Poisson bracket with the Hamiltonian both generate symmetries of the Hamiltonian and is a conserved quantity, which extends to quantum mechanics if Poisson brackets are replaced by commutators. For a Markov process, an observable commutes with the Hamiltonian iff both its expected value and that of its square are constant in time for every state.

The main results of the paper are the following three theorems.

Theorem. Let X be a finite set, $H : \mathbb{R}^X \to \mathbb{R}^X$ an infinitesimal stochastic operator, and O an observable. Then the following are equivalent:

1. $[O, H] = 0$.
2. $\frac{d}{dt}\left(f(O), \psi(t)\right) = 0$ for all polynomials $f : \mathbb{R} \to \mathbb{R}$ and all ψ obedient to the master equation with the Hamiltonian H.
3. $\frac{d}{dt}\left(O, \psi(t)\right) = \frac{d}{dt} f O^2, \psi(t) = 0$ for all ψ obedient to the master equation with the Hamiltonian H.
4. $O_i = O_j$ if i and j lie in the same connected component of the transition graph H.

Theorem. Suppose X is a σ-finite measure space and

$$U(t) : L^1(X) \to L^1(X)$$

is a Markov semigroup. Suppose O is an observable. Then

$$[O, U(t)] = 0$$

for all $t \geq 0$ iff for all probability distributions ψ on X, the expected values $(O, U(t)\psi)$ and $\left(O^2, U(t)\psi\right)$ are constant as a function of t.

Theorem. Suppose X is a σ-finite measure space and

$$U : L^1(X) \to L^1(X)$$

is a stochastic operator. Suppose O is an observable. Then $[O, U] = 0$ iff for all probability distributions ψ on X, we have

$$(O, U\psi) = (O, \psi)$$

and

$$\left(O^2, U\psi\right) = \left(O^2, \psi\right).$$

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

81P20 Stochastic mechanics (including stochastic electrodynamics)
81Q05 Closed and approximate solutions to the Schrödinger, Dirac, Klein-Gordon and other equations of quantum mechanics
47D07 Markov semigroups and applications to diffusion processes
60J25 Continuous-time Markov processes on general state spaces
70H33 Symmetries and conservation laws, reverse symmetries, invariant manifolds and their bifurcations, reduction for problems in Hamiltonian and Lagrangian mechanics

Full Text: DOI arXiv

References:

[1] Byers N., Israel Math. Conf. Proc. 12, in: The Heritage of Emmy Noether (1999)

[2] DOI: 10.1007/978-0-387-87868-3 · Zbl 1216.01011 · doi:10.1007/978-0-387-87868-3

[3] DOI: 10.1063/1.528145 · Zbl 0828.60037 · doi:10.1063/1.528145

[4] Stroock D., An Introduction to Markov Processes (2005) · Zbl 1068.60003

[5] DOI: 10.1007/s004400050091 · Zbl 0868.60064 · doi:10.1007/s004400050091

[6] Yosida K., Functional Analysis (1995) · Zbl 0830.46001