Fong, Brendan; **Sarazola, Maru**
A recipe for black box functors. (English) Zbl 07216044
Theory Appl. Categ. 35, 979-1011 (2020).

Network diagrams are often used to represent and reason about interconnected systems, having meaning in the relevant semantics of diagrams. Recently *hypergraph* categories are used to describe the algebraic structure of interconnected systems such as electric circuits, signal flow graphs, Markov processes, automata and so on [*J. C. Baez* and *B. Fong*, Theory Appl. Categ. 33, 1158–1222 (2018; Zbl 1402.18005); *F. Bonchi* et al., Inf. Comput. 252, 2–29 (2017; Zbl 1355.68188); Zbl *L. de Francesco Albasini* et al., Appl. Categ. Struct. 19, No. 1, 425–437 (2011; Zbl 1225.18006); *L. de Francesco Albasini* et al., RAIRO, Theor. Inform. Appl. 45, No. 1, 117–142 (2011; Zbl 1216.18005); *J. C. Baez* et al., J. Math. Phys. 57, No. 3, 033301, 30 p. (2016; Zbl 1336.60147); *A. Gianola* et al., LIPIcs – Leibniz Int. Proc. Inform. 72, Article 2, 17 p. (2017; Zbl 1433.68223)], where the syntax is represented by morphisms in a hypergraph category while the semantics is represented by another hypergraph category. Since semantics interpretation has the effect of wrapping the network in a black box, a hypergraph functor describing the semantics of a system is called a *black box functor*. This paper describes a general method for constructing such functors.

To extract from an open dynamical system a relation between input and output concentration in steady states, *J. C. Baez* and *B. S. Pollard* [Rev. Math. Phys. 29, No. 9, Article ID 1750028, 41 p. (2017; Zbl 1383.68053)] introduced a strong symmetric monoidal functor (called the black-boxing functor)

$$\blacksquare : \boldsymbol{Dynam} \to \boldsymbol{Rel}$$

where $\boldsymbol{Dynam}$ is a category of finite sets X as objects and cospans of finite sets $X \to N \leftarrow Y$ together with a suitably well-behaved vector field on N as morphisms, while $\boldsymbol{Rel}$ is the category of sets of binary relations. This paper provides a general recipe for constructing such functors, providing a streamlined proof of the functoriality of the fucntor $\blacksquare$, which is to be understood as a hypergraph functor.

Baez and Polland exploited what is known as a decorated cospans construction [*B. Fong*, Theory Appl. Categ. 30, 1096–1120 (2015; Zbl 1351.18003)] to define $\boldsymbol{Dynam}$. The main theme of this paper is a careful investigation of a generalization of this construction introduced in [*B. Fong*, Theory Appl. Categ. 33, 608–643 (2018; Zbl 1400.18011)] under the name of *decorated correlations*. Decorated cospans construct a hypergraph category from a finitely cocomplete category $\mathcal{C}$ and a lax symmetric monoidal functor

$$F : (\mathcal{C}, +) \to (\boldsymbol{Set}, \times)$$

Decorated correlations generalize this by also requiring a factorization sytem on $\mathcal{C}$, and extending F to a functor on a certain subcategory of $\boldsymbol{Cospan}\,(\mathcal{C})$.

The main results of this paper are the following two theorems (Theorem 3.25 and Theorem 5.4).

Theorem. The decorated correlations construction defines a functor

$$(-)\,\boldsymbol{Corel} : \boldsymbol{DecData} \to \boldsymbol{Hyp}$$

where $\boldsymbol{DecData}$ is the category of decorating data and $\boldsymbol{Hyp}$ is the category of hypergraph categories as objects and hypergraph functors as morphisms.

Theorem. The functor $(-)\,\boldsymbol{Corel}$ factors as the composite $\Phi \circ \boldsymbol{Kan}$, where these fucntors are part of adjunctions

$$\boldsymbol{Hyp} \underset{\Phi}{\overset{\boldsymbol{Alg}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \boldsymbol{CospanAlg} \underset{\boldsymbol{Kan}}{\overset{\iota}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \boldsymbol{DecData}$$

with $\boldsymbol{CospanAlg}$ being a full subcategory of $\boldsymbol{DecData}$ and the embedding $\iota : \boldsymbol{CospanAlg} \to$

DecData being of its left adjoint ***Kan***.

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MSC:

18M35 Categories of networks and processes, compositionality
18M30 String diagrams and graphical calculi
18B10 Categories of spans/cospans, relations, or partial maps

Keywords:

decorated corelation; Frobenius monoid; hypergraph category; black box functor; well-supported compact closed category

Full Text: Link

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