Carqueville, Nils; Meusburger, Catherine; Schaumann, Gregor
3-dimensional defect TQFTs and their tricategories. (English) Zbl 07173974
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An *n-dimensional topological quantum field theory* (TQFT) á la *M. Atiyah* [Publ. Math., Inst. Hautes Étud. Sci. 68, 175–186 (1988; Zbl 0692.53053); Turk. J. Math. 21, No. 1, 1–7 (1997; Zbl 0890.57019)] and *G. Segal* [Philos. Trans. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. 359, No. 1784, 1389–1398 (2001; Zbl 1041.81094)] is a symmetric monoidal functor from Bord_n to vector spaces, providing topological invariants of manifolds in particular. TQFT with defects can provide finer invariants and a far and away richer theory.

It is completely classical that 2-dimensional *closed* TQFTs are equivalent to commutative Frobenius algebras [*J. Kock*, Frobenius algebras and 2D topological quantum field theories. Cambridge: Cambridge University Press (2004; Zbl 1046.57001)]. By allowing labelled boundaries, one ascends to *open/closed* TQFTs $\mathcal{Z}^{\mathrm{oc}}$ [*C. I. Lazaroiu*, Nucl. Phys., B 603, No. 3, 497–530 (2001; Zbl 0983.81090)], which are known to be equivalent to a commutative Frobenius algebra (which $\mathcal{Z}^{\mathrm{oc}}$ assigns to $\boldsymbol{S}^1$) together with a *Calabi-Yau category* (whose Hom are what $\mathcal{Z}^{\mathrm{oc}}$ does to intervals with labelled endpoints) and certain relations between the two. As far as all known examples of open/closed TQFTs are concerned, it is only the Calabi-Yau category that matters, the Frobenius algebras being recovered as its Hochshild cohomology [*K. Costello*, Adv. Math. 210, No. 1, 165–214 (2007; Zbl 1171.14038)]. This means that the transition from 2-dimensional closed to open/closed TQFTs is from algebras to categories.

This naturally leads us to consider 2-dimensional defect TQFTs, of which closed and open/closed flavors are special cases. Every 2-dimensional defect TQFT naturally gives rise to pivotal 2-category [*A. Davydov* et al., Proc. Symp. Pure Math. 83, 71–128 (2011; Zbl 1272.57023)], subsuming B-twisted sigma models [*A. Căldăraru* and *S. Willerton*, New York J. Math. 16, 61–98 (2010; Zbl 1214.14013)], symplectic manifolds and Lagrangian correspondences [*K. Wehrheim*, Assoc. Women Math. Ser. 6, 3–90 (2016; Zbl 1396.57044)] and affine Landau-Ginzburg models [*N. Carqueville* and *D. Murfet*, Adv. Math. 289, 480–566 (2016; Zbl 1353.18004)].

The notion of defect TQFT generalizes to arbitrary dimension in principle. An n-dimensional defect TQFT should be a symmetric monoidal functor

$$\mathrm{Bord}_n^{\mathrm{def}} \to \mathrm{Vect}_k$$

where $\mathrm{Bord}_n^{\mathrm{def}}$ is some suitably augmented version of Bord_n allowing decorated submanifolds of various codimensions. Such TQFTs are expected to be described by n-categories with additional structure [*A. Kapustin*, in: Proceedings of the international congress of mathematicians (ICM 2010), Hyderabad, India, August 19–27, 2010. Vol. III: Invited lectures. Hackensack, NJ: World Scientific; New Delhi: Hindustan Book Agency. 2021–2043 (2011; Zbl 1233.57018)], but details have not been worked out for $n > 2$, nor even a precise definition of defect TQFT has appeared in the literature for $n > 2$. The principal objective in this paper is to remedy both issues for $n = 3$, paving the way to still higher dimensions.

Inspired constantly by Lurie's splendid work on the *cobordism hypothesis* [*J. Lurie*, in: Current developments in mathematics, 2008. Somerville, MA: International Press. 129–280 (2009; Zbl 1180.81122)], higher categories are predominant in *extended* TQFT, where decompositions along lower-dimensional subspaces promote Bord_n itself to a higher category. This includes the classification of fully extended 3-dimensional TQFTs valued in the $(\infty, 3)$-category of monoidal categories by spherical fusion categories [*C. L. Douglas* et al., "Dualizable tensor categories", Preprint, `arXiv:1312.7188`], the classification of 3-2-1 extended TQFTs valued in 2-Vect by modular tensor categories [*B. Barlett* et al., "Modular categories as representations of the 3-dimensional bordism 2-category", Preprint, `arXiv:1509.06811`], and the work on boundary conditions and surface defects [*J. Fuchs* et al., Commun. Math. Phys. 321, No. 2, 543–575 (2013; Zbl 1269.81169); *A. Kapustin* and *N. Saulina*, Proc. Symp. Pure Math. 83, 175–198 (2011; Zbl 1248.81206)]. Although extended TQFTs assume higher categories and higher functors from the outset, this paper contends that a defect TQFT is an ordinary symmetric monoidal functor, while everything leading to higher-categorical structures is contained in an augmented bordism category.

This paper consists of four sections. The authors claim that the most conceptual and economic way of systematically studying 3-dimensional defect TQFT is through the notion of defect cobordism, which is introduced in §2.1. Roughly speaking, defect bordisms are bordisms coming with a stratification whose 1-, 2- and 3-strata are decorated by a choice of label sets $\mathbb{D}$. Defect morphisms form the morphisms in a symmetric monoidal category $\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D})$, whose objects are compatibly decorated stratified closed surfaces. §3 defines a 3-dimensional defect TQFT to be a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_k$$

It is well known that every weak 2-category is equivalent to a strict 2-category, but the corresponding statement does not hold for the 3-dimensional case [*R. Gordon* et al., Coherence for tricategories. Providence, RI: American Mathematical Society (AMS) (1995; Zbl 0836.18001)]. As is reviewed in §3.1, the generically strictest version of a 3-category is what is called a *Gray category*. The following main result (Theorem 3.12 and Theorem 3.13) is established in §3.3 and §3.4.

Theorem. Every 3-dimensional defect TQFT $\mathcal{Z} : \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_k$ gives rise to a k-linear Gray category with duals $\mathcal{T}_{\mathcal{Z}}$.

The Gray category $\mathcal{T}_{\mathcal{Z}}$ is an invariant associated to $\mathcal{Z}$ measuring among other things how closed 3-dimensional TQFTs associated to unstratified bordisms are glued together. Roughly speaking, the idea behind the construction of $\mathcal{T}_{\mathcal{Z}}$ is to directly transport the decorated structure of bordisms in $\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D})$ into the graphical calculus for Gray categories with duals [*J. W. Barrett* et al., "Gray categories with duals and their diagrams", Preprint, `arXiv:1211.0529`].

§4 gives three classes examples of 3-dimensional defect TQFTs. In the first example the authors argue that even the trivial TQFT leads to an interesting tricategory, seeing how the trivial defect functor adds relations producing a 3-groupoid from the free pre-2-category $\mathcal{F}_d^p\mathbb{K}^{\mathbb{D}}$ associated to given defect data $\mathbb{D}$ (§4.1). In the second example the authors consider the Reshetikhin-Turaev theory [*N. Reshetikhin* and *V. G. Turaev*, Invent. Math. 103, No. 3, 547–597 (1991; Zbl 0725.57007)] associated to a modular tensor category $\mathcal{C}$, which can be regarded as a special class of defect TQFTs $\mathcal{Z}^{\mathcal{C}}$ with only 1-dimensional defects (§4.2). In the last subsection (§4.3) the authors exploit the work on homotopy quantum field theory [*V. Turaev* and *A. Virelizier*, Int. J. Math. 23, No. 9, 1250094, 28 p. (2012; Zbl 1254.57012)] to construct two classes of 3-dimensional defect TQFTs for every G-graded spherical fusion category, obtaining interesting surface defects which allows of extending Turaev-Viro theories to defect TQFTs.

The authors conclude §1 (introduction and summary) by providing five topics for future study.

1. It would be interesting to compare the approach in this paper with the correspondence between TQFTs and higher categories in [*S. Morrison* and *K. Walker*, Geom. Topol. 16, No. 3, 1481–1607 (2012; Zbl 1280.57026); Proc. Natl. Acad. Sci. USA 108, No. 20, 8139–8145 (2011; Zbl 1256.18007)].
2. The approach in this paper enables one to extend Reshetikhin-Turaev theory to produce not only invariants for 3-dimensional manifolds with embedded ribbons but also with embedded surfaces.
3. It was argued in [*A. Kapustin* et al., Nucl. Phys., B 816, No. 3, 295–355 (2009; Zbl 1194.81224); *A. Kapustin* and *L. Rozansky*, Commun. Number Theory Phys. 4, No. 3, 563–549 (2010; Zbl 1220.81169)] that Rozansky-Witten theory gives rise to a 3-category $\mathcal{T}^{\mathrm{RW}}$ bringing together algebraic and symplectic geometry. It would be interesting to comprehend $\mathcal{T}^{\mathrm{RW}}$ as a Gray category with duals.
4. There is a theory of orbifold completion of pivotal 2-categories [*N. Carqueville* and *I. Runkel*, Quantum Topol. 7, No. 2, 203–279 (2016; Zbl 1360.18007)] generalizing certain group actions on 2-dimensional TQFTs via the algebraic language of defects and leading to new equivalences of categories [*N. Carqueville* et al., J. Pure Appl. Algebra 220, No. 2, 759–781 (2016; Zbl 1333.18004); *N. Carqueville* and *A. Quintero Velez*, "Calabi-Yau completions and orbifold equivalences", Preprint, `arXiv:1509.00880`]. The idea of orbifold completion applies to TQFTs of any dimension n and should be developed for $n = 3$, which would bring new TQFTs.
5. 3-dimensional TQFTs have applications in quantum computing, where orbifoldable group actions have received much attention. The approach in this paper would offer a robust and conceptual framework for generalizations of these constructions.

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