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Quantum $\mathfrak{gl}_{1|1}$ and tangle Floer homology. (English) Zbl 07066539
Adv. Math. 350, 130-189 (2019).

The Reshetikhin-Turaev construction [Zbl 0725.57007] is a machinery turning a representation W of a quantized enveloping algebra $U_q(\mathfrak{g})$ into a tangel invariant. This includes the Jones polynomial as the case of $\mathfrak{g} = \mathfrak{sl}_2$ and $W = U$ (the vector representation) and the Alexander polynomial as the case of $\mathfrak{g} = \mathfrak{gl}_{1|1}$ and $W = V$ (the vector representation).

As interesting as these invariants are, still more interesting are their categorifications of more complicated invariants. Khovanov homology is a poster child for categorification [Zbl 1275.17012; Zbl 1205.17010; Zbl 1261.17006; Zbl 1243.17004; Zbl 1321.57031; Zbl 1002.57006]. More recently, the papers [Zbl 1188.81117; Zbl 1214.81113] seeked to categorify an even wider swath of quanatum algebra including quantized enveloping algebras, tensor products of their integrable highest weight representations and Reshetikhin-Turaev intertwiners. The paper [Zbl 07000045] has used this approach to construct link homology theories categorifying the Reshetikhin-Turaev for all respresentations and Kac-Moody types. None of these extend to the case of Alexander polynomial, while the unique categorification of $U_q(\mathfrak{gl}_{1|1})$ [Zbl 1309.81113; Zbl 1305.18053; Zbl 1355.57006] is by no means of representation-theoretic character.

Knot Floer homology was introduced in [Zbl 1062.57019, arXiv:math/0306378], associating a bigraded chain complex $\widetilde{\mathrm{CFK}}(\mathcal{H})$ to a Heegaard diagram $\mathcal{H}$ for a link L. The homology $\widetilde{\mathrm{HFK}}(L)$ of $\widetilde{\mathrm{CFK}}(\mathcal{H})$ is an invariant of L. Although $\widetilde{\mathrm{CFK}}(\mathcal{H})$ is of a completely combinatorial description [Zbl 1179.57022], the invariant is still global in nature, and local modifications are only partly understood [Zbl 1203.57012; Zbl 1161.57005; Zbl 1062.57019]. In order to fit $\widetilde{\mathrm{HFK}}(L)$ into the general pattern of Reshetikhin-Turaev invariants, two hurdles should be addressed, namely, locality and the relation to $U_q(\mathfrak{gl}_{1|1})$.

The second and third authors of this paper [Zbl 1366.57005] have introduced a local construction of knot Floer homology. A dg algebra $A(P)$ is associated to each oriented 0-maniifold P, and a dg bimodule $\widetilde{\mathrm{CT}}(\mathcal{T})$ is associated to each oriented tangle $\mathcal{T}$. The general structure should appear familiar to any bordered Heegaard Floer homologist. The bimodules in question are type DA structures in the sense of [Zbl 1315.57036], and composition of these bimodules is via the box tensor product.

The Alexander polynomial, then, admits a categorification with local pieces very much like its construction in the Reshetikhin-Turaev invariant. The principal objective in this paper is to demonstrate that these local pieces categorify Reshetikhin-Turaev counterparts. The main results are the following two theorems, the first being established in §4.1 and §4.2 while the second being proved in §4.3.

In §2.3, to a sign sequence $P \in \{\pm 1\}^n$, the authors associate the $U_q(\mathfrak{gl}_{1|1})$-representation $V_P \otimes L(\lambda_P)$, where V_P is a tensor product of copies V and V^*, and $L(\lambda_P)$ is an appropriately chosen 2-dimensional representation depending on P and a basis B whose vectors are in bijection with subsets of $[n] = \{0, 1, \dots, n\}$. The dg algebra $A(P)$ has primitive idempotents in bijection with subsets of $[n]$, $e_{\boldsymbol{s}}$ standing for the primitive idempotent in correspondence with $\boldsymbol{s} \subseteq [n]$.

Theorem. Let
$$P = (P_1, \dots, P_n) \in \{\pm 1\}^n$$
be a sign sequence. Then the Grothendieck group of *dg* modules over the *dg* algebra $A(P)$ is a free $\mathbb{Z}[q^{\pm 1}]$-module with basis
$$\{[A(P)e_{\boldsymbol{s}}] \mid \boldsymbol{s} \subseteq [n]\}$$
Identifying the basis vector $[A(P)e_{\boldsymbol{s}}]$ with the basis vector in B associated to the subset $\boldsymbol{s}$ determines an isomorphism of vector spaces
$$K_0(A(P)) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{C}(q) \cong V_P \otimes L(\lambda_P)$$
Let $\mathcal{T}$ be a tangle and color each strand of $\mathcal{T}$ by the vector representation V. Under the above identification, up to an overall factor of a positive integer power of $(1 - q^{-2})$, box tensor product with the type DA bimodule $\widetilde{\mathrm{CT}}(\mathcal{T})$ acts on $K_0(A(P))$ as the Reshetikhin-Turaev intertwiner associated to the colored

tangle $\mathcal{T}$ (with reversed orientation) tensored with $\mathrm{id}_{L(\lambda_P)}$.

The authors introduce dg bimodules $E(P)$ and $F(P)$ over $(A(P), A(P))$ which act on $K_0(A(P))$ as the elements E and F of $U_q(\mathfrak{gl}_{1|1})$. Distinct from other categorifications of quantized enveloping algebras, these dg bimodules do not arise by induction and restriction with respect to a tower of algebras comprising the dg algebras $A(P)$.

Theorem. For any sign sequence P, under the identification of the elementary basis with the basis B in the previous theorem, the actions the dg bimodules $E(P)$ and $F(P)$ on $K_0(A(P))$ equal the actions of $E, F \in U_q(\mathfrak{gl}_{1|1})$ on $V_P \otimes L(\lambda_P)$. There are quasi-isomorphisms

$$E(P)\widetilde{\otimes}_{A(P)}E(P) \simeq 0$$
$$F(P)\widetilde{\otimes}_{A(P)}F(P) \simeq 0$$

Furthermore, there exists a distinguished triangle

$$E(P)\widetilde{\otimes}_{A(P)}F(P) \to A(P) \to F(P)\widetilde{\otimes}_{A(P)}E(P) \to E(P)\widetilde{\otimes}_{A(P)}F(P)[1]$$

For any tangle $\mathcal{T}$, we have

$$E(-\partial^0\mathcal{T}) \boxtimes \widetilde{\mathrm{CT}}(\mathcal{T}) \simeq A(-\partial^0\mathcal{T}) \boxtimes \widetilde{\mathrm{CT}}(\mathcal{T})\widetilde{\otimes}_{A(\partial^1\mathcal{T})}E(\partial^1\mathcal{T})$$
$$F(-\partial^0\mathcal{T}) \boxtimes \widetilde{\mathrm{CT}}(\mathcal{T}) \simeq A(-\partial^0\mathcal{T}) \boxtimes \widetilde{\mathrm{CT}}(\mathcal{T})\widetilde{\otimes}_{A(\partial^1\mathcal{T})}F(\partial^1\mathcal{T})$$

as type AA bimodules over $(A(-\partial^0\mathcal{T}), A(\partial^1\mathcal{T}))$.

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