Bunke, Ulrich; Engel, Alexander; Kasprowski, Daniel; Winges, Christoph
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If $\mathcal{C}$ is a categor and W is a set of morphisms in $\mathcal{C}$, then the localization functor

$$\ell : \mathcal{C} \to \mathcal{C}_\infty := \mathcal{C}\left[W^{-1}\right]$$

in ∞-categories can be considered [*J. Lurie*, "Higher algebra", `http://people.math.harvard.edu/~lurie/papers/HA.pdf`, Definition 1.3.4.1; *D.-C. Cisinski*, Higher categories and homotopical algebra. Cambridge: Cambridge University Press (2019; Zbl 1430.18001), Definition 7.1.2], where we consider $\mathcal{C}$ as an ∞-category by its nerve. If the relative category $(\mathcal{C}, W)$ is extendable to a simplicial model category in which all objects are cofibrant, then we have an equivalence of ∞-categories

$$\mathcal{C}_\infty \simeq \mathrm{N}^{coh}(\mathcal{C}^{cf})$$

where the right-hand side is the nerve of the simplicial category of cofibrant-fibrant objects of $\mathcal{C}$ [`http://people.math.harvard.edu/~lurie/papers/HA.pdf`, Definition 1.3.4.15 and Theorem 1.3.4.20]. This explicit description of $\mathcal{C}_\infty$ is often very helpful in calculating mapping spaces in $\mathcal{C}_\infty$ or in identifying limits or colmits of diagrams in $\mathcal{C}_\infty$.

This paper consists of three chapters. Chapter 1 is an introduction. Chapter 2 consists of 4 sections. §2.1 introduces categories of marked categories, marked preadditive categories and additive categories, various relations between these categories being given by forgetful functors and their adjoints. Their enrichments in groupoids and simplicial sets are described. §2.2 describes model category structures on the categories $\mathbf{Cat}$, $\mathbf{Cat}^+$, $\mathbf{preAdd}$ and $\mathbf{PreAdd}^+$. The first main result of the paper is presented here.

Theorem 2.2.2. Let $\mathcal{C}$ be $\mathbf{Cat}$ or $\mathbf{preAdd}$. The simplicial category $\mathcal{C}$ (or $\mathcal{C}^+$)with weak equivalences, cofibrations and fibrations depicted in the following is a simplicial and combinatorial model category.

[1] A morphism $f : \mathbf{A} \to \mathbf{B}$ in $\mathcal{C}$ (or $\mathcal{C}^+$) is a weak equivalence iff it admits an inverse $g : \mathbf{B} \to \mathbf{A}$ up to isomorphisms (or marked isomorphisms);

[2] A morphism in $\mathcal{C}$ (or $\mathcal{C}^+$)is a cofibration iff it is injective on objects;

[3] A morphism in $\mathcal{C}$ (or $\mathcal{C}^+$)is a fibration iff it has the right lifting property for trivial cofibrations.

§2.3 introduces Bousfield localizations of $\mathbf{preAdd}$ and $\mathbf{PreAdd}^+$ whose categories of fibrant objects are exactly the additive categories or marked additive categories. §2.4 introduces the ∞-category of (marked) preadditive categories and that of (marked)additive categories.

$$\mathbf{preAdd}_\infty^{(+)} := \mathbf{preAdd}^{(+)}\left[W^{-1}_{\mathbf{preAdd}^{(+)}}\right]$$
$$\mathbf{Add}_\infty^{(+)} := \mathbf{preAdd}^{(+)}\left[W^{-1}_{\mathbf{Add}^{(+)}}\right]$$

where $W_{\mathbf{preAdd}^{(+)}}$ denotes the weak equivalences in $\mathbf{preAdd}^{(+)}$, and $W_{\mathbf{Add}^{(+)}}$ denotes the weak equivalences in the Bousfield localization $L_{\{v,w\}}\mathbf{preAdd}^{(+)}$. Two adjunctions are presented.

$$\mathbf{preAdd}_\infty^{+} \leftrightarrows \mathbf{preAdd}_\infty$$
$$\mathbf{preAdd}_\infty^{(+)} \leftrightarrows \mathbf{Add}_\infty^{(+)}$$

It is shown that

Proposition 2.4.6. Let $\mathcal{C}$ be $\mathbf{Cat}^{(+)}$, $\mathbf{Add}^{(+)}$ or $\mathbf{preAdd}^{(+)}$, which is to be considered as a category enriched in groupoids and therefore as a strict $(2,1)$-category, denoted by $\mathcal{C}_{(2,1)}$. We first apply the usual nerve functor to the morphism categories of $\mathcal{C}_{(2,1)}$ to obtain a category enriched in Kan complexes. We then apply the coherent nerve functor to get a quasi-category $\mathrm{N}_2(\mathcal{C}_{(2,1)})$. The obvious functor

$$\mathrm{N}(\mathcal{C}_{(1,1)}) \to \mathrm{N}_2(\mathcal{C}_{(2,1)})$$

sends equivalences to equivalences, therefore descending to a functor

$$\mathcal{C}_\infty \to \mathrm{N}_2(\mathcal{C}_{(2,1)})$$

which is claimed to be an equivalence.

The results in Chapter 2 are applied in Chapter 3 to get the other two main results of the paper.

Theorem 3.3.1. We have a natural equivalence

$$\operatorname*{colim}_{BG} \ell_{\mathbf{preAdd}^{(+)},BG}(\underline{\mathbf{A}}) \simeq \ell_{\mathbf{preAdd}^{(+)}}(\mathbf{A}\sharp BG)$$

Theorem 3.4.3. We have a natural equivalence

$$\lim_{BG} \ell_{\mathbf{preAdd}^{(+)},BG}(\underline{\mathbf{A}}) \simeq \ell_{\mathbf{preAdd}^{(+)}}(\widehat{\mathbf{A}}^G)$$

Reviewer: Hirokazu Nishimura (Tsukuba)

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