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Braided Cartan calculi and submanifold algebras. (English) Zbl 07190559

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On the one hand, noncommutative calculi based on *derivations* rather than generalizations of differential forms, in which differential forms are dual objects to derivations, are investigated in [*A. Cap* et al., Acta Math. Univ. Comen., New Ser. 62, No. 1, 17–49 (1993; Zbl 0830.58002); *M. Dubois-Violette* and *P. W. Michor*, J. Geom. Phys. 20, No. 2–3, 218–232 (1996; Zbl 0867.53023); *P. Schupp*, in: Quantum groups and their applications in physics. Amsterdam: IOS Press; Tokyo: Ohmsha. 507–524 (1996; Zbl 0866.17011); *P. Schupp* et al., in: Proceedings of the XXIIth international conference on differential geometric methods in theoretical physics, Ixtapa-Zihuatanejo, México, September 20-24, 1993. México: Universidad Nacional Autónoma de México. 125–134 (1994; Zbl 0846.17019)], though bimodules are to be considered over the center of the algebra. On the other hand, *S. L. Woronowicz* [Commun. Math. Phys. 122, No. 1, 125–170 (1989; Zbl 0751.58042)] generalized the notion of *Cartan calculus* on quantum groups, the crucial ingrediant being given by the de Rham differential to be understood as a linear map $d : H \to \Gamma$ from a Hopf algebra H to a bicovariant H-bimodule Γ abiding by the Leibniz rule

$$d(ab) = (da)b + a(db)$$

for all $a, b \in H$. It is shown that such a first-order calculus admits an extension to the exterior algebra.

This paper proposes an intermediate path, sticking to derivation-based calculi while admitting a Hopf algebra symmetry in order to avoid central bimodules. The author was inspired by *twisted Cartan calculi* coming from Drinfel'd's twist and accompanied by twisted covariant derivatives and metrics as a generalization of classical Riemannian geometry [*P. Aschieri* et al., Classical Quantum Gravity 23, No. 6, 1883–1911 (2006; Zbl 1091.83022)]. The additional braided symmetries therein were the main inspiration inviting the author to consider noncommutative Cartan calculi which depends only on a triangular structure rather than on the Drinfel'd's twist itself. The appropriate categorical framework for this attempt was provided in [*G. E. Barnes* et al., J. Geom. Phys. 89, 111–152 (2015; Zbl 1321.83035); ibid. 106, 234–255 (2016; Zbl 1342.83220)], where the category of equivariant braided symmetric bimodules with respect to a triangular Hopf algebra and a braided commutative algebra is symmetric braided and monoidal with respect to the tensor product over the algebra. The author generalizes the algebraic construction of the Cartan calculus to this category to get what is called the *braided Cartan calculus*, vector fields being represented by the braided Lie algebra of braided derivations and multivector fields being a braided Gerstenhaber algebra while differential forms constituting a braided Grassmann algebra. From a categorical standpoint, a Drinfel'd's twist corresponds to a functor, and its action is to be put down as a gauge equivalence on the symmetric braided monoidal category [*P. Aschieri* and *A. Schenkel*, Adv. Theor. Math. Phys. 18, No. 3, 513–612 (2014; Zbl 1317.14008); *C. Kassel*, Quantum groups. New York, NY: Springer-Verlag (1995; Zbl 0808.17003)]. It is shown that the Drinfel'd's functor respects the braided Cartan calculus in the sense that it intertwines the braided Lie derivative, insertion, de Rham differential and Schouten-Nijenhuis bracket. The author also generalizes covariant derivatives and metrics to braided symmetric setting.

We should remark that the classical Cartan calculus and the twisted Cartan calculus are to be regarded as braided Cartan calculus, the former being one with respect to any cocommutative Hopf algebra with trivial triangular structure and the latter being one with respect to the twisted Hopf algebra, triangular structure and algebra. The braided Cartan calculus on submanifold algebras is investigated as an application, and it is shown that they are projected from the ambient algebra in accordance to gauge equivalences. The author suggests, as future topics, to generalize the braided Cartan calculus to the setting of [*G. E. Barnes* et al., J. Geom. Phys. 89, 111–152 (2015; Zbl 1321.83035)], to Lie-Rinehart algebras [*J. Huebschmann*, Ann. Inst. Fourier 48, No. 2, 425–440 (1998; Zbl 0973.17027)], and furthermore to Hochschild cohomology and the Cartan calculus in [*B. Tsygan*, Clay Math. Proc. 16, 19–66 (2012; Zbl 1316.18001); *D. Tamarkin* and *B. Tsygan*, Methods Funct. Anal. Topol. 6, No. 2, 85–100 (2000; Zbl 0965.58010)].

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MSC:

13B Commutative ring extensions and related topics
18M15 Braided monoidal categories and ribbon categories
58A15 Exterior differential systems (Cartan theory)

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Hopf algebra; Drinfel'd twist; noncommutative geometry; submanifold algebra; braided commutative geometry; noncommutative Cartan calculus

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