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Estimates by gap potentials of free homotopy decompositions of critical Sobolev maps.

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The *free homotopy decomposition* appears in the construction of harmonic and polyharmonic maps being known in many examples to generate all the homotopy classes through free homotopy decomposition (Theorem 14 in [A. Gastel and A. J. Nerf, *Calc. Var. Partial Differ. Equ.* 47, No. 3–4, 499–521 (2013; [Zbl 1276.58003](#))], Theorem 5.5 in [J. Sacks and K. Uhlenbeck, *Ann. Math.* (2) 113, 1–24 (1981; [Zbl 0462.58014](#))]). The free homotopy decomposition is an invariant under homotopies of the maps, but is not in general a faithful invariant. The author shows that maps of the same free homotopy decomposition abide by the same fractional Sobolev bound (Theorem 1.2).

Theorem. Let $m \in \mathbb{N}$ and \mathcal{N} a compact Riemannian manifold. If $f \in \mathcal{C}(\mathbb{S}^m, \mathcal{N})$ has a free homotopy decomposition into $f_1, \dots, f_k \in \mathcal{C}(\mathbb{S}^m, \mathcal{N})$, then for every $s \in (0, 1]$ and every $p \in [m, +\infty)$ with $p = m/s > 1$, one has

$$\inf \{ \mathcal{E}^{s,p}(g) \mid g \in (\mathcal{C} \cap W^{s,p})(\mathbb{S}^m, \mathcal{N}) \text{ is homotopic to } f \} \leq \sum_{i=1}^k \mathcal{E}^{s,p}(f_i)$$

The proof of the above theorem is performed by gluing together the maps f_1, \dots, f_k with an arbitrarily small energetic cost of gluing through conformal transformations by Mercator projections.

By taking the above theorem into account, it was established that for every $\lambda > 0$, there exists a finite set \mathcal{F} and $k \in \mathbb{N}$ such that every map $f \in (\mathcal{C} \cap W^{s,p})(\mathbb{S}^m, \mathcal{N})$ pursuant to $\mathcal{E}^{s,m/s}(f) \leq \lambda$ has a free homotopy decomposition into k maps of the set \mathcal{F}

- in case of $m = 1, s = \frac{1}{2}, p = 2$ [E. Kuwert, *J. Reine Angew. Math.* 505, 1–22 (1998; [Zbl 0933.58014](#))]
- in case of $m \geq 1, s = 1$ [F. Duzaar and E. Kuwert, *Calc. Var. Partial Differ. Equ.* 6, No. 4, 285–313 (1998; [Zbl 0909.49008](#)), Theorem 4]
- in case of $m \geq 1, s = 1 - \frac{1}{m+1}$ [T. Müller, *Manuscr. Math.* 103, No. 4, 513–540 (2000; [Zbl 0981.49025](#)), Theorem 5.1]
- in case of $m = 2, s = 1$ [R. Schoen and J. Wolfson, *J. Differ. Geom.* 58, No. 1, 1–86 (2001; [Zbl 1052.53056](#)), Lemma 5.2]

The critical case $sp = m$ for estimates is to be seen as a limiting case between the classical continuous picture of homotopy classes in the supercritical $sp > m$ and the combination of collapses and appearances of homotopy classes in the subcritical case $sp < m$ (B. White [J. Differ. Geom. 23, 127–142 (1986; [Zbl 0588.58017](#))], H. Brezis and Y. Li [C. R. Acad. Sci., Paris, Sér. I, Math. 331, No. 5, 365–370 (2000; [Zbl 0972.46014](#)); J. Funct. Anal. 183, No. 2, 321–369 (2001; [Zbl 1001.46019](#))], F. Hang and F. Lin [Discrete Contin. Dyn. Syst. 13, No. 5, 1097–1124 (2005; [Zbl 1093.46017](#)); Acta Math. 191, No. 1, 55–107 (2003; [Zbl 1061.46032](#)); Commun. Pure Appl. Math. 56, No. 10, 1383–1415 (2003; [Zbl 1038.46026](#)); Math. Res. Lett. 8, No. 3, 321–330 (2001; [Zbl 1049.46018](#))]).

The main result of the paper, claiming that the above estimates are in fact consequences of a stronger gap potential estimate, goes as follows:

Theorem. Let $m \in \mathbb{N}$ and \mathcal{N} a compact Riemannian manifold. If $\varepsilon > 0$ is small enough, then there is a constant $C > 0$ such that for every $\lambda > 0$, there exists a finite set $\mathcal{F}^\lambda \subset \mathcal{C}(\mathbb{S}^m, \mathcal{N})$ with any f pursuant to the inequality

$$\iint_{\substack{(x,y) \in \mathbb{S}^m \times \mathbb{S}^m \\ d_{\mathcal{N}}(f(y), f(x)) > \varepsilon}} \frac{1}{|y-x|^{2m}} dy dx \leq \lambda$$

being of a free homotopy decomposition into f_1, \dots, f_k with $k \leq C\lambda$.

The above theorem describes sharply the homotopy classes that can be encountered under a boundedness assumption on the double integral. The proof of the above theorem goes within a geometric setting where the sphere \mathbb{S}^m is put down as the boundary at infinity of the hyperbolic space \mathbb{H}^{m+1} , and the manifold \mathcal{N} is embedded isometrically into a Euclidean space \mathbb{R}^v . The extension of the map f by averaging at each point $x \in \mathbb{H}^{m+1}$ over the sphere at infinity provides a Lipschitz-continuous extension $F : \mathbb{H}^{m+1} \rightarrow \mathbb{R}^v$. The set on which the values of the map F cannot be retracted to \mathcal{N} is contained in a number of balls whose diameter and number is controlled with allowing to construct the families of maps by the classical Ascoli compactness argument for continuous maps.

The appearance of free homotopy decomposition in which the way of gluing the maps is uncontrolled is to be thought of as topological bubbling phenomenon, which is a topological version of the geometric bubbling phenomenon in conformally invariant geometric problems [O. Druet et al., Blow-up theory for elliptic PDEs in Riemannian geometry. Princeton, NJ: Princeton University Press (2004; [Zbl 1059.58017](#)); T. H. Parker, J. Differ. Geom. 44, No. 3, 595–633 (1996; [Zbl 0874.58012](#)); J. Sacks and K. Uhlenbeck, Ann. Math. (2) 113, 1–24 (1981; [Zbl 0462.58014](#))]. In many cases, however, the above main result implies that maps abiding by a bound on the gap potential can only belong to finitely many homotopy classes (Theorem 1.4). In the one-dimensional case, the total variation of the maps occurring in the decomposition can be estimated (Theorem 1.5).

Reviewer: [Hirokazu Nishimura \(Tsukuba\)](#)

MSC:

- [58C](#) Calculus on manifolds; nonlinear operators
- [46E35](#) Sobolev spaces and other spaces of “smooth” functions, embedding theorems, trace theorems
- [55M25](#) Degree, winding number
- [55P99](#) Homotopy theory
- [55Q25](#) Hopf invariants
- [58A12](#) de Rham theory in global analysis

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