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L^1-Poincaré inequalities for differential forms on Euclidean spaces and Heisenberg groups.

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The celebrated *Sobolev inequality*, called (p-Sobolev), claims that for every $1 \leq p < n$, there exists a constant $C(n,p)$ such that all smooth compactly supported functions u on $\mathbb{R}^n$ abide by

$$\|u\|_q \leq C(n,p)\,\|\nabla u\|_p$$

provided that $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. The celebrated *Poincaré inequality*, called (p-Poincaré), claims that there exists a constant c_u such that

$$\|u - c_u\|_q \leq C(n,p)\,\|\nabla u\|_p$$

provided that $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$.

The principal objective in this paper is to generalize these inequalities to differential forms. The question considered here is to ask whether, given a closed differential h-form ϖ in $L^p(\mathbb{R}^n)$, there exists an $(h-1)$-form ϕ in $L^q(\mathbb{R}^n)$ with $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$ such that $d\phi = \varpi$ and

$$\|\phi\|_q \leq C(n,p,h)\,\|\varpi\|_p$$

This paper establishes (1-Poincaré) for h-forms of degree $h < n$ in de Rham complex $(\Omega^\bullet, d)$, relying on *L. Lanzani* and *E. M. Stein*'s observation [Math. Res. Lett. 12, No. 1, 57–61 (2005; Zbl 1113.26015)] that the duality estimate underlying *J. Bourgain* and *H. Brezis*' result [J. Eur. Math. Soc. (JEMS) 9, No. 2, 277–315 (2007; Zbl 1176.35061)] descends $(n-1)$-forms to forms of lower degrees, and the resulting Gagliardo and Nirenberg inequalities [*L. Nirenberg*, Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. 13, 115–162 (1959; Zbl 0088.07601); C.I.M.E., Principio di Minimo e sue Applicazioni alle Equazioni funzionali 1–48 (1960; Zbl 0108.10001); *E. Gagliardo*, Ric. Mat. 8, 24–51 (1959; Zbl 0199.44701)]. The approach generalizes to the non-commutative Heisenberg groups $\mathbb{H}^n$ endowed with [*M. Rumin*'s complex [J. Differ. Geom. 39, No. 2, 281–330 (1994; Zbl 0973.53524)] $(E_0^\bullet, d_c)$.

As for Heisenberg groups, one can exploit Lanzani and Stein's type arguments [*A. Baldi* and *B. Franchi*, J. Funct. Anal. 265, No. 10, 2388–2419 (2013; Zbl 1282.35391); *A. Baldi* et al., Matematiche 75, No. 1, 167–194 (2020; Zbl 07175593)], relying on *S. Chanillo* and *J. Van Schaftingen*'s inequality [Math. Res. Lett. 16, No. 2–3, 487–501 (2009; Zbl 1184.35119)]. The main results, namely global Poincaré and Sobolev inequalities (Theorem 1.1) and their local verions (Theorem 1.2), are stated for both Euclidean and Heisenberg cases. Smoothing homotopies on Riemannian or contact subRiemannian manifolds of bounded geometry are constructed.

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MSC:

58A10 Differential forms in global analysis
35R03 PDEs on Heisenberg groups, Lie groups, Carnot groups, etc.
43A80 Analysis on other specific Lie groups
53D10 Contact manifolds, general
46E35 Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems

Keywords:

Heisenberg groups; differential forms; Sobolev-Poincaré inequalities; contact manifolds; homotopy formula

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References:

[1] Baldi, A.; Barnabei, M.; Franchi, B., A recursive basis for primitive forms in symplectic spaces and applications to Heisenberg groups, Acta Math. Sin. Engl. Ser., 32, 3, 265-285 (2016), MR 3456421 · Zbl 1381.53149

[2] Baldi, A.; Franchi, B., Sharp a priori estimates for div-curl systems in Heisenberg groups, J. Funct. Anal., 265, 10, 2388-2419 (2013), MR 3091819 · Zbl 1282.35391

[3] Baldi, A.; Franchi, B.; Pansu, P., Gagliardo-Nirenberg inequalities for differential forms in Heisenberg groups, Math. Ann., 365, 3-4, 1633-1667 (2016), MR 3521101 · Zbl 1359.58002

[4] Baldi, A.; Franchi, B.; Pansu, P., Poincaré and Sobolev inequalities for differential forms in Heisenberg groups and contact manifolds

[5] Baldi, A.; Franchi, B.; Pansu, P., \(L^1\)-Poincaré and Sobolev inequalities for differential forms in Euclidean spaces, Sci. China Math., 62, 6, 1029-1040 (2019) · Zbl 1417.58001

[6] Baldi, A.; Franchi, B.; Tchou, N.; Tesi, M. C., Compensated compactness for differential forms in Carnot groups and applications, Adv. Math., 223, 5, 1555-1607 (2010) · Zbl 1184.43007

[7] Baldi, A.; Franchi, B.; Tesi, M. C., Hypoellipticity, fundamental solution and Liouville type theorem for matrix-valued differential operators in Carnot groups, J. Eur. Math. Soc., 11, 4, 777-798 (2009) · Zbl 1184.35117

[8] Benilan, P.; Brezis, H.; Crandall, M. G., A semilinear equation in \(L^1(R^N)\), Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4), 2, 4, 523-555 (1975), MR 0390473 · Zbl 0314.35077

[9] Bergh, J.; Löfström, J., Interpolation Spaces. An Introduction, Grundlehren der Mathematischen Wissenschaften, vol. 223 (1976), Springer-Verlag: Springer-Verlag Berlin-New York, MR 0482275 · Zbl 0344.46071

[10] Bourgain, J.; Brezis, H., New estimates for elliptic equations and Hodge type systems, J. Eur. Math. Soc., 9, 2, 277-315 (2007), MR 2293957 (2009h:35062) · Zbl 1176.35061

[11] Capogna, L.; Garofalo, N., Boundary behavior of nonnegative solutions of subelliptic equations in NTA domains for Carnot-Carathéodory metrics, J. Fourier Anal. Appl., 4, 4-5, 403-432 (1998), MR 1658616 · Zbl 0926.35043

[12] Capogna, L.; Garofalo, N.; Nhieu, D.-M., Examples of uniform and NTA domains in Carnot groups, (Proceedings on Analysis and Geometry. Proceedings on Analysis and Geometry, Novosibirsk Akademgorodok, 1999. Proceedings on Analysis and Geometry. Proceedings on Analysis and Geometry, Novosibirsk Akademgorodok, 1999, Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk (2000)), 103-121, (Russian). MR 1847513 · Zbl 1017.30020

[13] Chanillo, S.; Van Schaftingen, J., Subelliptic Bourgain-Brezis estimates on groups, Math. Res. Lett., 16, 3, 487-501 (2009), MR 2511628 (2010f:35042) · Zbl 1184.35119

[14] Curcă, E., On the representation as exterior differentials of closed forms with \(L^1\)-coefficients, C. R. Math. Acad. Sci. Paris, 357, 4, 355-359 (2019), MR 3955206 · Zbl 1416.58001

[15] de Rham, G., Variétés différentiables. Formes, courants, formes harmoniques, Actualités Sci. Ind., vol. 1222 (1955), Publ. Inst. Math. Univ. Nancago III, Hermann et Cie: Publ. Inst. Math. Univ. Nancago III, Hermann et Cie Paris, MR 0068889 · Zbl 0065.32401

[16] Evans, L. C.; Gariepy, R. F., Measure Theory and Fine Properties of Functions, Textbooks in Mathematics (2015), CRC Press: CRC Press Boca Raton, FL, MR 3409135 · Zbl 1310.28001

[17] Federer, H.; Fleming, W. H., Normal and integral currents, Ann. Math. (2), 72, 458-520 (1960), MR 0123260 · Zbl 0187.31301

[18] Folland, G. B., Subelliptic estimates and function spaces on nilpotent Lie groups, Ark. Mat., 13, 2, 161-207 (1975), MR MR0494315 (58 #13215) · Zbl 0312.35026

[19] Folland, G. B.; Stein, E. M., Hardy Spaces on Homogeneous Groups, Mathematical Notes, vol. 28 (1982), Princeton University Press: Princeton University Press Princeton, N.J., MR MR657581 (84h:43027) · Zbl 0508.42025

[20] Franchi, B.; Gallot, S.; Wheeden, R. L., Sobolev and isoperimetric inequalities for degenerate metrics, Math. Ann., 300, 4, 557-571 (1994), MR 1314734 (96a:46066) · Zbl 0830.46027

[21] Franchi, B.; Lu, G.; Wheeden, R. L., Representation formulas and weighted Poincaré inequalities for Hörmander vector fields, Ann. Inst. Fourier (Grenoble), 45, 2, 577-604 (1995), MR 1343563 (96i:46037) · Zbl 0820.46026

[22] Franchi, B.; Penso, V.; Serapioni, R., Remarks on Lipschitz domains in Carnot groups, (Geometric Control Theory and Sub-Riemannian Geometry. Geometric Control Theory and Sub-Riemannian Geometry, Springer INdAM Ser., vol. 5 (2014), Springer: Springer Cham), 153-166, MR 3205101 · Zbl 1305.53041

[23] Franchi, B.; Serapioni, R.; Serra Cassano, F., Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields, Houst. J. Math., 22, 4, 859-890 (1996), MR 1437714 · Zbl 0876.49014

[24] Garofalo, N.; Nhieu, D.-M., Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, Commun. Pure Appl. Math., 49, 10, 1081-1144 (1996), MR 1404326 (97i:58032) · Zbl 0880.35032

[25] Gromov, M., Carnot-Carathéodory Spaces Seen from Within, Sub-Riemannian Geometry, Progr. Math., vol. 144, 79-323 (1996), Birkhäuser: Birkhäuser Basel, MR MR1421823 (2000f:53034) · Zbl 0864.53025

[26] Helffer, B.; Nourrigat, J., Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs, Progress in Mathematics, vol. 58 (1985), Birkhäuser Boston Inc.: Birkhäuser Boston Inc. Boston, MA, MR MR897103 (88i:35029) · Zbl 0549.35026

[27] Iwaniec, T.; Lutoborski, A., Integral estimates for null Lagrangians, Arch. Ration. Mech. Anal., 125, 1, 25-79 (1993), MR MR1241286 (95c:58054) · Zbl 0793.58002

[28] Lanzani, L.; Stein, E. M., A note on div curl inequalities, Math. Res. Lett., 12, 1, 57-61 (2005), MR 2122730

(2005m:58001) · Zbl 1113.26015

[29] Long, R. L.; Sheng Nie, F., Weighted Sobolev inequality and eigenvalue estimates of Schrödinger operators, (Harmonic Analysis. Harmonic Analysis, Tianjin, 1988. Harmonic Analysis. Harmonic Analysis, Tianjin, 1988, Lecture Notes in Math., vol. 1494 (1991), Springer: Springer Berlin), 131-141, MR 1187073 · Zbl 0786.46034

[30] Lu, G., Polynomials, higher order Sobolev extension theorems and interpolation inequalities on weighted Folland-Stein spaces on stratified groups, Acta Math. Sin. Engl. Ser., 16, 3, 405-444 (2000), MR 1787096 · Zbl 0973.46020

[31] Mattila, P., Fractals and rectifiability, (Geometry of Sets and Measures in Euclidean Spaces. Geometry of Sets and Measures in Euclidean Spaces, Cambridge Studies in Advanced Mathematics, vol. 44 (1995), Cambridge University Press: Cambridge University Press Cambridge), MR 1333890

[32] McDuff, D.; Salamon, D., Introduction to Symplectic Topology, Oxford Mathematical Monographs (1998), The Clarendon Press, Oxford University Press: The Clarendon Press, Oxford University Press New York, MR 1698616 · Zbl 1066.53137

[33] Mitrea, D.; Mitrea, M.; Monniaux, S., The Poisson problem for the exterior derivative operator with Dirichlet boundary condition in nonsmooth domains, Commun. Pure Appl. Anal., 7, 6, 1295-1333 (2008), MR 2425010 · Zbl 1157.58007

[34] Montgomery, R., A Tour of Subriemannian Geometries, Their Geodesics and Applications, Mathematical Surveys and Monographs, vol. 91 (2002), American Mathematical Society: American Mathematical Society Providence, RI, MR MR1867362 (2002m:53045) · Zbl 1044.53022

[35] Monti, R., Distances, boundaries and surface measures in Carnot-Carathéodory spaces (2001), School of Sciences, University of Trento: School of Sciences, University of Trento Italy, Ph.D. thesis

[36] Monti, R.; Morbidelli, D., Regular domains in homogeneous groups, Trans. Am. Math. Soc., 357, 8, 2975-3011 (2005), MR 2135732 · Zbl 1067.43003

[37] Nhieu, D.-M., The Neumann problem for sub-Laplacians on Carnot groups and the extension theorem for Sobolev spaces, Ann. Mat. Pura Appl. (4), 180, 1, 1-25 (2001), MR 1848049 · Zbl 1052.46026

[38] Ornstein, D., A non-equality for differential operators in the \(L_1\) norm, Arch. Ration. Mech. Anal., 11, 40-49 (1962), MR 0149331 · Zbl 0106.29602

[39] P. Pansu, Cup-products in \(l^{q , p}\)-cohomology: discretization and quasi-isometry invariance, Preprint, 2017.

[40] Pansu, P.; Tripaldi, F., Averages and the \(\ell^{q , 1}\) cohomology of Heisenberg groups, Ann. Math. Blaise Pascal, 26, 1, 81-100 (2019) · Zbl 07162632

[41] Rumin, M., Formes différentielles sur les variétés de contact, J. Differ. Geom., 39, 2, 281-330 (1994), MR MR1267892 (95g:58221) · Zbl 0973.53524

[42] Rumin, M., An introduction to spectral and differential geometry in Carnot-Carathéodory spaces, Rend. Circ. Mat. Palermo Suppl. (2), 75, 139-196 (2005), MR MR2152359 (2006g:58053) · Zbl 1105.58012

[43] Saka, K., Besov spaces and Sobolev spaces on a nilpotent Lie group, Tohoku Math. J. (2), 31, 4, 383-437 (1979), MR 558675 · Zbl 0429.43004

[44] Schwarz, G., Hodge Decomposition—a Method for Solving Boundary Value Problems, Lecture Notes in Mathematics, vol. 1607 (1995), Springer-Verlag: Springer-Verlag Berlin, MR MR1367287 (96k:58222) · Zbl 0828.58002

[45] Stein, E. M., Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, (Monographs in Harmonic Analysis, III. Monographs in Harmonic Analysis, III, Princeton Mathematical Series, vol. 43 (1993), Princeton University Press: Princeton University Press Princeton, NJ), With the assistance of Timothy S. Murphy. MR MR1232192 (95c:42002) · Zbl 0821.42001

[46] Van Schaftingen, J., Limiting Bourgain-Brezis estimates for systems of linear differential equations: theme and variations, J. Fixed Point Theory Appl., 15, 2, 273-297 (2014), MR 3298002 · Zbl 1311.35005

[47] Varopoulos, N. Th.; Saloff-Coste, L.; Coulhon, T., Analysis and Geometry on Groups, Cambridge Tracts in Mathematics, vol. 100 (1992), Cambridge University Press: Cambridge University Press Cambridge, MR MR1218884 (95f:43008) · Zbl 0813.22003

[48] Vodop'yanov, S. K.; Greshnov, A. V., On the continuation of functions of bounded mean oscillation on spaces of homogeneous type with intrinsic metric, Sib. Mat. Zh., 36, 5, 1015-1048 (1995), i. MR 1373594