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Local homotopy theory. (English) Zbl 1320.18001

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"Local homotopy theory is the study of diagrams of spaces or spectrum-like objects and weak equivalences, where the weak equivalences are determined by a topology on the underlying index category." This book, concerned with local homotopy theory, is divided into 4 parts, namely, the first part (Chapters 2 and 3) entitled "Preliminaries", the second part (Chapters 4–7) entitled "Simplicial Presheaves and Simplicial Sheaves", the third part (Chapters 8 and 9) entitled "Sheaf Cohomology Theory" and the fourth part (Chapters 10 and 11) entitled "Stable Homotopy Theory".

Chapter 5 deals with constructions of the basic model structures for simplicial presheaves and simplicial sheaves, including the injective model structures for both kinds of categories. The first main result (Theorem 5.8) in §5.1 claiming that the category of simplicial presheaves on a small Grothendieck site has what is called an injective model structure, as well as the second main result (Theorem 5.9) in the section claiming that the category of simplicial sheaves on a small Grothendieck site is also of an injective model structure which is Quillen equivalent to the above one is of accumulating applications in such arenas as algebraic K-theory, algebraic geometry, number theory and algebraic topology. A plethora of model structures between the above one and the local projective model structure of [*B. A. Blander*, K-Theory 24, No. 3, 283–301 (2001; Zbl 1073.14517)] appearing in [*J. F. Jardine*, Can. Math. Bull. 49, No. 3, 407–413 (2006; Zbl 1107.18007)], which have been used in complex analytic geometry, are dealt with in §5.5. The main result (Theorem 5.49) in §5.6, due to [*G. Biedermann*, Homology Homotopy Appl. 10, No. 1, 305–325 (2008; Zbl 1138.55015)], gives the n-equivalence model structure for simplicial presheaves. The descent concept, which is an extention of gluing in topology, is discussed in the context of homotopy theory in §5.4, where two of the main descent theorem from algebraic K-theory are established, namely, the Brown-Gersten descent theorem for the Zariski topology (Theorem 5.33) in [*K. S. Brown* and *S. M. Gersten*, Lect. Notes Math. 341, 266–292 (1973; Zbl 0291.18017)] and the Morel-Voevodsky descent theorem for the Nisnevich topology (Theorem 5.39) in [*F. Morel* and *V. Voevodsky*, Publ. Math., Inst. Hautes Étud. Sci. 90, 45–143 (1999; Zbl 0983.14007)], both of which form the basis in the description of the theory of stacks in Chapter 9.

Chapters 2–4 provide the essential requisites for Theorems 5.8 and 5.9. Chapter 2 gives a succinct description of the homotopy theory of simplicial sets and a first take on the homotopy theory of diagrams of simplicial sets (§2.1). The first main result in §2.2 is Theorem 2.13 claiming that there is a model structure for simplicial sets, in which the cofibrations are monomorphisms and the weak equivalences are depicted via topological realization. The second main result in §2.2 is Theorem 2.19 claiming that the fibrations in the above model structure are Kan fibrations, whose topological realizations are well known to be Serre fibrations. The proof of Theorem 2.19 is merely sketchy, while the proof of Theorem 2.13 exploits the bounded monomorphism property for simplicial sets (Lemma 2.16) which recurs in various guises throughout the book. Much more detail on the homotopy theory of simplicial sets can be found in the author's [*P. G. Goerss* and *J. F. Jardine*, Simplicial homotopy theory. Basel: Birkhäuser (1999; Zbl 0949.55001)]. The main result in §2.3 (Proposition 2.22) is concerned with the model structure for diagrams of simplicial sets, which was first observed in [*A. K. Bousfield* and *D. M. Kan*, Homotopy limits, completions and localizations. Berlin-Heidelberg-New York: Springer-Verlag (1972; Zbl 0259.55004)]. Chapter 3 is a brief introduction to topos theory in geometric flavor, dealing mainly with Grothendieck toposes. Theorem 3.31 (*M. Barr*'s theorem [J. Pure Appl. Algebra 5, 265–280 (1974; Zbl 0294.18009)]), commonly called the principle of Boolean localization, is the most significant result of topos theory that finds applications in homotopy theory. The final section of Chapter 3 is devoted to the proof of this theorem, following a letter of Joyal to Grothendieck in 1984 and [*D. H. van Osdol*, Am. J. Math. 99, 1193–1204 (1977; Zbl 0374.18010)]. The chapter includes also a proof of Giraud's theorem (Theorem 3.17). Chapter 4 is concerned with a description of the interplay between local fibrations and local weak equivalences for simplicial presheaves and simplicial sheaves with a view of the model structure results coming later.

The main result in Chapter 6 is the Verdier hypercovering theorem (Theorem 6.12) of the form in [*F. Morel* and *V. Voevodsky*, Publ. Math., Inst. Hautes Étud. Sci. 90, 45–143 (1999; Zbl 0983.14007)], which follows readily from the generalized Verdier hypercovering theorem (Theorem 6.5) claiming that two cocycles are in the same path component if and only if they represent the same morphism in the homotopy category and that every morphism in the homotopy category is to be represented by a cocycle.

Chapter 7 starts with a model category $\mathbf{M}$ abiding by a list of assumptions met in all examples of interest, together with a functor $L : \mathbf{M} \to \mathbf{M}$ obedient to a shorter list of assumptions including preservation of weak equivalences. The main result in the chapter is Theorem 7.5 claiming that there is a model structure on the category underlying $\mathbf{M}$ whose cofibrations are the same as those in the original model structure and whose weak equivalences are the L-equivalences. This general setup gives the basis for all localized homotopy theories in the book, including the traditional f-localization for a cofibration $f : A \to B$, f-localization of presheaves of chain complexes (§8.5) and all stable homotopy theories of Chapters 10 and 11.

The subject in Chapter 8 is the homotopy-theoretic treatment of the homological algebra of abelian sheaves and presheaves. The first main result in the chapter is Theorem 8.6 claiming that the existence of injective model structures for the categories of presheaves and sheaves of simplicial R-modules together with their Quillen equivalence, which follows from Theorem 5.9. The most interesting point in the proof of the theorem is that free abelian group functor takes local weak equivalences of simplicial presheaves to local weak equivalences of simplicial abelian presheaves. "The classical relation between chain complexes and simplicial abelian groups is somewhat deceptive, however, in that the Dold-Kan correspondence induces a Quillen equivalence between the model structure on simplicial abelian groups, which is induced from the standard model structure on simplicial sets, and the naive model structure on ordinary chain complexes from basic homological algebra." The preservation of local weak equivalences by the free R-module functor gives rise to a Quillen adjunction, by which sheaf cohomology with morphisms in the simplicial presheaf homotopy category is identified in Theorem 8.26 (the second main result in the chapter).

Chapter 9, concerning non-abelian cohomology, consists of 5 sections. Higher stack theory as the local homotopy theory of presheaves of groupoids enriched in simplicial sets is described in §9.3 and §9.4. The first main result (Theorem 9.30) in §9.3 is concerned with what is called the Dwyer-Kan model structure for groupoids enriched in simplicial sets [*W. G. Dwyer* and *D. M. Kan*, Indag. Math. 46, 379–385 (1984; Zbl 0559.55023)]. The second main result (Theorem 9.43) in §9.3 gives rise to a model structure on groupoids enriched in simplicial sets, where the fibrations (respectively, weak equivalences) are the maps inducing Kan fibrations (respectively, weak equivalences) of simplicial sets by the Eilenberg-MacLane construction. This model structure is Quillen equivalent to the standard model structure on simplicial sets via the loop groupoid functor as the left ajoint of the Eilenberg-MacLane functor. The model structure in Theorem 9.43 is elevated to a local model structure for presheaves of groupoids enriched in simplicial sets in Theorem 9.50, which is Quillen equivalent to the injective model structure on simplicial presheaves. This is the main result in §9.4. The model structure in Theorem 9.50 is specialed to a homotopy theory of n-types in Theorem 9.56 and to a homotopy theory for presheaves of 2-groupoids in Theorem 9.57. The final section (§9.5) describes the relationship between presheaves of groupoids and presheaves of 2-groupoids on a small Grothendieck site. The basic idea is that 2-groupoids can be used to classify various classes of groupoids up to weak equivalences, which dates back to as far as [*J. Giraud*, Cohomologie non abelienne. Berlin-Heidelberg-New York: Springer-Verlag (1971; Zbl 0226.14011)] in the early 1970s.

The concluding two chapters (Chapters 10 and 11) give an account of the fundamentals of local stable homotopy theory, including the stable homotopy theory of presheaves of spectra on arbitrary sites and the various forms of motivic stable homotopy theory. The stable homotopy theory of presheaves of spectra has fertile applications in algebraic K-theory and, most recently, in traditional stable homotopy theory via the theory of topological modular forms. Motivic stable homotopy theory and motivic cohomology theory together give the setting for many of the recent advances in calculation in the arena of algebraic K-theory, including [*V. Voevodsky*, Ann. Math. (2) 174, No. 1, 401–438 (2011; Zbl 1236.14026)] and [*A. Suslin* and *V. Voevodsky*, NATO ASI Ser., Ser. C, Math. Phys. Sci. 548, 117–189 (2000; Zbl 1005.19001)]. There are multiple styles of local stable homotopy theory of varying degrees of complexity, all of which are exquisitely modelled upon the motivic stable homotopy theory [*J. F. Jardine*, Doc. Math., J. DMV 5, 445–553 (2000; Zbl 0969.19004)]. The localization-theoretic construction of the motivic stable model structure in Chapter 10 differs considerably from that in [Zbl 0969.19004], which used the method in [*A. K. Bousfield* and *E. M. Friedlander*, Lect. Notes Math. 658, 80–130 (1978; Zbl 0405.55021)], and resembles the method in [*M. Hovey*, J. Pure Appl. Algebra 165, No. 1, 63–127 (2001; Zbl 1008.55006)].

In Chapter 10, abstracting the major features of motivic stable homotopy theory, the author gives f-local stable homotopy theory of presheaves of T-spectra on an arbitrary site, where f is a cofibration of simplicial presheaves and T is an arbitrary parameter object. The same machinery continues to work in the study of symmetric T-spectra in f-local settings in Chapter 11.

The book is well written, except that the author consistently refers to references by natural numbers which should have been increased by one (e.g., Heller [37] should have been Heller [38]).

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MSC:

18-02	Research exposition (monographs, survey articles) pertaining to category theory
18G55	Nonabelian homotopical algebra (MSC2010)
18G30	Simplicial sets; simplicial objects in a category (MSC2010)
18G50	Nonabelian homological algebra (category-theoretic aspects)
55Q10	Stable homotopy groups

Keywords:

local homotopy theory; stack; simplicial sheaf; simplicial presheaf; injective model structure; projective model structure; Grothendieck topos; n-type; descent; algebraic K-theory; Morel-Voevodsky descent theorem; Brown-Gersten descent theorem; algebraic geometry; topos theory; Barr's theorem; Giraud's theorem; the principle of Boolean localization; Verdier-hypercovering theorem; stable homotopy theory; localization; cohomology theory; Quillen equivalence; Dold-Kan correspondence; Dwyer-Kan model structure; Eilenberg-MacLane construction; Kan fibration; 2-groupoid; Friedlander-Milnor conjector; Bloch-Kato conjecture; Lichtenbaum-Quillen conjecture; spectra; gerbe

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