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Categories and sheaves. (English) Zbl 1118.18001

Grundlehren der Mathematischen Wissenschaften 332. Berlin: Springer (ISBN 3-540-27949-0/hbk). x, 497 p. (2006).

Category theory, initiated by Eilenberg and MacLane in the fourties of the previous century, may be seen as part of a wider movement transcending mathematics, of which structuralism, originating with the famous anthropologist Lévi-Strauss in the fifties of the previous century and then rampant in various academic arenas, is presumably another facet. Since Grothendieck's revolution in algebraic geometry in the middle of the previous century, it has spread so rapidly that it now occupies the language of mathematics, replacing the classical notions of sets and functions.

Sheaf theory provides a tool for passing from local data to global ones, and a great deal of mathematics and physics can not dispense with it. The category-theoretical essence of sheaf theory is nowadays grasped as topos theory.

Homological algebra first arose as a convenient language for describing topological properties in algebraic terms in the 1940s. Its first phase ended in 1956 with the appearance of Cartan and Eilenberg's famous monograph "Homological Algebra". Its second phase began with Grothendieck's long paper "Sur quelques points d'algèbre homologique" in 1957 and was always dominated by the influence of his school of algebraic geometry. The third phase, still running, began with Verdier's thesis written under Grothendieck's direction and slowly spread beyond the confines of algebraic geometry. It is characterized by its ever-increasing use of derived categories and triangulated categories. The machinery was taken up by M. Sato and his school on microlocal analysis, and has since been used in the theory of $\mathcal{D}$-modules and perverse sheaves with applications to representation theory, in which both authors of this monograph are famous.

The aim of this monograph is to describe these three branches together, which will undoubtedly prepare the reader for the same authors' "Sheaves on Manifolds" published in the same series [Grundlehren der Mathematischen Wissenschaften, 292. Berlin etc.: Springer-Verlag (1990; Zbl 0709.18001)]. Indeed the monograph under review may be put down as a detailed account of the first two chapters of their previous one, in which their treatment was surely hasty.

The monograph is divided into three parts. The general theory of categories and functors is presented in Chapters 1-7. The most important result in the first chapter is Yoneda lemma, which enables us to embed any category $\mathcal{C}$ into the category $\mathcal{C}^\wedge$ of $\mathbf{S}et$-valued functors on $\mathcal{C}$. This naturally leads to the notion of a representable functor. The succeeding two chapters are devoted to projective and inductive limits, which are so fundamental that they are comparable with unions and intersections in set theory. The fourth chapter is devoted to a rapid treatment of tensor categories, which were introduced as an axiomazation of tensor products of vector spaces and are nowadays seen in various areas, in particular, in quantum groups and knot theory. In Chapter 5 various criteria are given on the representability of $\mathbf{S}et$-valued functors. Chapters 6 and 7 are devoted to ind-objects and localization respectively. Chapter 6 is very valuable, for there is no other concise treatment on ind-objects written in English. Localization is an essential step in constructing derived categories, and the classical reference is Gabriel and Zisman's monograph "Calculus of Fractions and Homotopy Theory" published in 1967.

The second part of the monograph consists of 8 chapters, dealing with homological algebra. Chapter 8 is devoted to additive and abelian categories, establishing in particular the Gabriel-Popescu theorem that a Grothendieck category may be embedded into the category of modules over the ring of endomorphisms of a generator. In Chapter 9 many results on filtrant inductive limits are extended to the case of π-filtrant inductive limits for an infinite cardinal π. In Chapter 10 the authors treat triangulated categories with their localization and the construction of cohomological functors in detail. Chapter 11 is devoted to the main idea of homological algebra to replace an object in an additive category $\mathcal{C}$ by a complex of objects in $\mathcal{C}$, the components of which have good properties. In the next chapter the additive category $\mathcal{C}$ is assumed to be abelian, in which one can consider the j-th cohomological functor H^j of a complex. With these preparations the authors can give an easy way to the construction of the derived category of an abelian category in Chapter 13. In Chapter 14 the unbounded derived categories of Grothendieck categories are studied. The final chapter of the second part is devoted to the derived category of the category of ind-objects of an abelian category $\mathcal{C}$.

The final part of the monograph is concerned with sheaves, consisting of four chapters. In Chapter 16 the authors present Grothendieck topologies axiomatically and then introduce the notions of local epimorphisms and local isomorphisms. The next chapter is devoted to sheaves on Grothendieck topologies. In Chapter 18 sheaves of $\mathcal{R}$-modules are introduced, where $\mathcal{R}$ is a sheaf of rings on a site X. In this situation the tools of Chapter 14 are applicable. The final chapter is devoted to a sketchy treatment of stacks and twisted sheaves.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18-02 Research exposition (monographs, survey articles) pertaining to category theory

18F20 Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)

14F05 Sheaves, derived categories of sheaves, etc. (MSC2010)

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