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Categories for quantum theory. An introduction. (English) Zbl 07081551
Oxford Graduate Texts in Mathematics 28. Oxford: Oxford University Press (ISBN 978-0-19-873962-3/hbk; 978-0-19-873961-6/pbk). xii, 324 p. (2019).

Monoidal category theory serves as a powerful platform for explicating logical aspects of quantum theory, yielding an abstract language for parallel and sequential composition and helping understand many high-level quantum phenomena. This book, consisting of nine chapters (from 0 through 8) introduces the reader to categorical quantum mechanics with an intuitive graphical calculus in front. The book stems from a mini-course by the authors at a spring school in 2010, aimed at beginning graduate students from various fields. The first notes written in 2012 formed a basis for a graduate course at the Department of Computer Science Science in Oxford, which has run every year since. The text formed the basis for postgraduate summer and winter schools in Dalhousie, Pisa and Palmse as well. The book strikes a good balance between theory and applications, a bit biased in favor of theory, though there are so many applications as to force the authors to only hint at them. Applications are interspersed throughout the main text and exercises, while the theory aims at maximum reasonable generality. The exposition is pleasant and truly convincing.

A synopsis of the book goes as follows:

Chapter 0 is a brief introduction to category theory (§0.1), linear algebra and the theory of Hilbert spaces (§0.2) and quantum theory (§0.3), being intended to make the book as self-contained as possible.

Chapter 1 introduces the theory of *monoidal categories* [*S. MacLane*, Rice Univ. Stud. 49, No. 4, 28–46 (1963; Zbl 0244.18008); *J. Benabou*, C. R. Acad. Sci., Paris 256, 1887–1890 (1963; Zbl 0111.02201); *J. Benabou*, C. R. Acad. Sci., Paris 260, 752–755 (1965; Zbl 0192.10901)] with a lot of examples. §1.1 shows how categories like the category ***Rel*** of sets and relations, the category ***Hilb*** of Hilbert spaces and the category ***Set*** of sets and mappings can be given a monoidal structure. §1.2 introduces a visual notation called the *graphical calculus*, the correctness of which is based upon *coherence theorem* to be established in §1.3, where the strictification and coherence theorems [*D. B. A. Epstein*, Invent. Math. 1, 221–228 (1966; Zbl 0146.02502; *S. MacLane*, Rice Univ. Stud. 49, No. 4, 28–46 (1963; Zbl 0244.18008)] are established.

Chapter 2, consisting of four sections, shows that many aspects of linear algebra is describable within the categorical setting. §2.1 examines the abstraction of the base field. Lemma 2.3 claiming that the scalars are commutative in a monoidal category was proved in 1980 in a distinct guise [*G, M. Kelly* et al., J. Pure Appl. Algebra 19, 193–213 (1980; Zbl 0447.18005)]. The realization that endomorphisms of the tensor unit behave as scalars was explicated in [*S. Abramsky* and *B. Coecke*, "A categorical semantics of quantum protocols", Log. Comput. Sci. 19, 415—425 (2004; Zbl 1151.81002]. §2.2 deals with the addition of vectors. §2.3 shows that inner products can be described abstractly using a *dagger*, a contravariant involutive endofunctor on the category that is compatible with the monoidal structure. The systematic exploitation of daggers in the way of the book started with [*P. Selinger*, Selinger, Peter (ed.), Proceedings of the 4th international workshop on quantum programming languages (QPL 2006), Oxford, UK, 17–19 July 2006. Amsterdam: Elsevier. Electronic Notes in Theoretical Computer Science 210, 107–122 (2008)Zbl 1279.18006]. Self-duality in the form of involutive endofunctors on categories traces back to as early as 1950 [*S. MacLane*, Bull. Am. Math. Soc. 56, 485–516 (1950)Zbl 0045.29905]. §2.4 demonstrates how to use the techniques in the previous sections to model significant feature of quantum mechanics such as classical data, superposition and measurement.

Chapter 3, consisting of four sections, investigates entanglement in terms of monoidal categories, using the notion of dual object and building up to the important notion of *compact category*. The use of compact categories in foundations of quantum mechanics was initiated in [*S. Abramsky* and *B. Coecke*, loc. cit.]. §3.1 introduces the basic definition and establishes its basic properties. §3.2 is concerned with the quantum teleportation ptotocol. §3.3 shows that the presence of dual objects ensures that tensor products interact well with any linear structure available. §3.4 makes a fairly thorough study of various ways dual objects on different objects cooperate. Abstract traces in monoidal categories discussed in the last section (Definition 3.59) was introduced in [*A, Joyal* et al., Math. Proc. Camb. Philos. Soc. 119, No. 3, 447–468 (1996)Zbl 0845.18005]. Any compact category is a so-called traced monoidal category. It was established in [*M. Hasegawa*, Math. Struct. Comput. Sci. 19, No. 2, 217–244 (2009; Zbl 1165.18007)] that abstract traces in a compact category are unique. Conversely, any traced monoidal category gives rise to a compact category by what is called the Int-construction.

Chapter 4, consisting of three sections, is concerned with *monoids* and *comonoids*, both of them are introduced in §4.1. §4.2 establishes the *no-deleting* and *no-cloning* theorems. The no-cloning theorem was established independently in [Nature 299, 802–803 (1982); Phys. Lett. A 92, 309–311 (1982)]. Its categorical version in Theorem 4.27 was in [Zbl 1192.81013], in which the no-deleting theorem (Theorem 4.20) due to Coecke was presented. Theorem 4.28 presented in §4.3 was given in [*Z. Petrić*. Stud. Log. 70, No. 2, 271–296 (2002; Zbl 1022.03050)].

Chapter 5, consisting of six sections, is concerned with *Frobenius structures* of a monoid and comonoid interacting in accordance with what is called the *Frobenius law*. §5.1 investigates its basic consequences. The Frobenius law is named after *F. Georg Frobenius* [Berl. Ber. 1903, 504-537, 634–645 (1903; JFM 34.0238.02)]. It was *T. Nakayama* [Ann. Math., Princeton, (2) 42, 1–21 (1941; JFM 67.0092.04); Ann. Math. (2) 42, 1–21 (1941; Zbl 0026.05801)] who coined the name. The formulation with multiplication and comultiplication in the book goes back to *F. W. Lawvere* [Semin. Triples categor. Homology Theory, ETH 1966/67, Lect. Notes Math. 80, 141–155 (1969; Zbl 0165.03204)]. *A. Carboni* and *R. F. C. Walters* [J. Pure Appl. Algebra 49, 11–32 (1987; Zbl 0637.18003); *A. Carboni* et al., Cah. Topol. Géom. Différ. Catég. 46, No. 3, 187–188 (2005; Zbl 1074.18505); Theory Appl. Categ. 19, Spec. Vol. CT2006 Conf., 93–124 (2007; Zbl 1146.18300)] used the formulation to axiomatize bicategories of relations. *Dijkgaaf* [A geometrical approach to two-dimensional Conformal Field Theory, Ph.D. thesis, University of Utrecht, 1989] established that the category of commutative Frobenius structures is equivalent to to that of two-dimensional topological quantum field theories. A comprehensive book on Frobenius strcuctures is available [*J. Kock*, London Mathematical Society Student Texts 59. Cambridge: Cambridge University Press (ISBN 0-521-54031-3/pbk; 0-521-83267-5/hbk). xiii, 240 p. (2004; Zbl 1046.57001)]. §5.2 is concerned with normal forms, establishing the *noncommutative spider theorem* (Theorem 5.21) and the *commutative spider theorem* (Theorem 5.22), the latter of which was due to *L. Abrams* [Frobenius Algebra Structures in Topological Quantum Field Theory and Quantum Cohomology, Ph.D., Johns Hopkins University, 1997] with a category-theoretic proof ascribed to [*S, Lack*, Theory Appl. Categ. 13, 147–163 (2004; Zbl 1062.18007)]. §5.3 establishes that the Frobenius law obtains precisely when the Cayley embedding in Proposition 4.13 preserves some structure. §5.4 classifies the special dagger Frobenius structures in the category ***FHilb*** of finite-dimensional Hilbert spaces [*B. Coecke*, Math. Struct. Comput. Sci. 23, No. 3, 555–567 (2013; Zbl 1276.46016)] and the category ***Rel*** of sets and relations [*D. Pavlovic*, Bruza, Peter (ed.) et al., Quantum interaction. Third international symposium, QI 2009, Saarbrücken, Germany, March 25–27, 2009. Proceedings. Berlin: Springer (ISBN 978-3-642-00833-7/pbk). Lecture Notes in Computer Science 5494. Lecture Notes in Artificial Intelligence, 143–157 (2009; Zbl 1229.68039); *C. Heunen* et al., J. Pure Appl. Algebra 217, No. 1, 114–124 (2013; Zbl 1271.18004)] in terms of operator algebras and groupoids respectively, with the commutative case being of special interest, as in ***FHilb*** this corresponds to a choice of orthonormal basis (Theorem 5.36 [Zbl 1276.46016]).

Chapter 6, consisting of five sections, considers what happens when two Frobenius structures interact, specifically, when they are *maximally incompatible* or *complementary*. §6.1 gives a definition making sense in arbitrary monoidal dagger category [*B. Coecke* et al., Aceto, Luca (ed.) et al., Automata, languages and programming. 35th international colloquium, ICALP 2008, Reykjavik, Iceland, July 7–11, 2008. Proceedings, Part II. Berlin: Springer (ISBN 978-3-540-70582-6/pbk). Lecture Notes in Computer Science 5126, 298-310 (2008; Zbl 1155.81316)], seeing that it comes down to the standard notion of mutually unbiased bases from quantum information theory in the category ***Hilb*** of Hilbert spaces, classifying the complementary groupoids in the category ***Rel*** of sets and relations and characterizing complementarity in terms of a canonical morphism being unitary. §6.2 addresses the *Deutsch-Jozsa algorithm* [*J. Vicary*; Proceedings of the 2013 28th annual ACM/IEEE symposium on logic in computer science, LICS 2013, Tulane University, New Orleans, LA, USA, June 25–28, 2013. Los Alamitos, CA: IEEE Computer Society (ISBN 978-0-7695-5020-6). 93–102 (2013; Zbl 1366.68067)]. §6.3 links complementarity to Hopf algebras, finding out that this well-investigated notion gives rise to a stronger form of complementarity. Turning to quantum computation, §6.4 discusses how many *qubit* gates are to be modelled in categorical quantum mechanics using merely complementary Frobenius structures, such as controlled negation, controlled phase gates and arbitrary single qubit gates. §6.5 finally discusses the *ZX calculus* [textitB, Coecke, Aceto, Luca (ed.) et al., Automata, languages and programming. 35th international colloquium, ICALP 2008, Reykjavik, Iceland, July 7–11, 2008. Proceedings, Part II. Berlin: Springer (ISBN 978-3-540-70582-6/pbk). Lecture Notes in Computer Science 5126, 298–310 (2008; Zbl 1155.81316)], which is a sound and complete way to handle quantum computations using only equations in graphical calculus.

Chapter 7, consisting of six sections, revolves around *completely positive maps*. §7.1 investigates evolution of mixed states of systems, finding out that evolutions correspond to completely positive maps and mixed states are no other than completely positive maps from the tensor unit to a system. §7.2 describes the main construction of the chapter, starting with the category of pure states to get the corresponding category of mixed states. §7.3 considers completely positive maps to and from classical structures, seeing that the subcategory of classical structures and completely positive maps models statistical mechancis. §7.4 studies the subcategory of completely positive maps between *completely noncommutative* Frobenius structures in the sense that every observable commuting with all others must be trivial. Returning to the CP construction, §7.5 extends the axiomatization of categories of quantum structures using *environment structures* to an axiomatization of any category of the form $\mathrm{CP}[\boldsymbol{C}^{\mathrm{pure}}]$, enabling one to discuss quantum teleportation for mixed states by using the relationship between objects of $\mathrm{CP}[\boldsymbol{C}^{\mathrm{pure}}]$ and Frobenius structures in $\boldsymbol{C}^{\mathrm{pure}}$. §7.6 investigates how the CP construction interacts with biproducts, much like in §3.3, finding out that if $\boldsymbol{C}$ has dagger biproducts, then so does $\mathrm{CP}[\boldsymbol{C}]$ [*Chris Heunen, Aleks Kissinger* and *Peter Selinger*, "Completely positive projections and biproducts", Electron. Proc. Theor. Comput. Sci. 171, 71–83 (2014)].

Chapter 8, consisting of three sections, sketches higher categories. §8.1 introduces *monoidal* 2-*categories* and their graphical calculus based on surfaces, investigating duality in *monoidal* 2-*categories* and seeing how the theory of commutative dagger Frobenius structures emerges from this in an elegant way. §8.2 introduces 2-*Hilbert spaces* and investigate their properties. §8.3 studies quantum teleportation and quantum dense coding from a higher-categorical perspective by using these techniques.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

81-02 Research exposition (monographs, survey articles) pertaining to quantum theory
81P05 General and philosophical questions in quantum theory
81P16 Quantum state spaces, operational and probabilistic concepts
18D10 Monoidal, symmetric monoidal and braided categories (MSC2010)
18D15 Closed categories (closed monoidal and Cartesian closed categories, etc.)
18D20 Enriched categories (over closed or monoidal categories)
81P68 Quantum computation
46L07 Operator spaces and completely bounded maps

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