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Quantum Riemannian geometry. (English) Zbl 07114689
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Noncommutative geometry arose from experience with quantum theory. By the 1920s, Dirac had already speculated about geometry with noncommutative x, p coordinates. Indeed, matrix algebras were the first examples of noncommutative algebras, and quantum mechanics was once called matrix mechanics. The great theorems of Gel'fand and Naimark for C*-algebras and the GNS construction of Hilbert space representations of noncommutative C*-algebras were driven by quantum theory. K-theory, universal differentials, projective modules and connections for noncommutative algebras, together with Hochschild and cyclic coomology, were a natural progression in this direction, culminating in Connes' celebrated notion of a spectral triple as an abstract *Dirac operator* in the early 1980s. The mid 1980s witnessed the emergence of large classes of noncommutative algebras as part of the *quantum group revolution*, these objects having arisen on the one hand as generalized symmetries in quantum intergrable systems (the Drinfeld-Jimbo quantum groups $U_q(\mathfrak{g})$) and on the other hand from ideas of quantum Born reciprocity or observable-state duality in quantum gravity [Non-commutative-geometric groups by a bicrossproduct construction: Hopf algebras at the Planck scale, the second author, PhD thesis, Harvard University, 1988]. The first class of quantum groups is of direct interst in various branches of mathematics from category theory to knot theory, while the second class is of particular significance as Poincaré quantum symmetries of quantum spacetime. Quantum groups of both kinds stand to noncommutative geometry in such a way as classical Lie groups stand to classical differential geometry.

The book, consisting of nine chapters, is concerned with noncommutative geometry. One of the principles of the book is to include both the pure mathematical background and applications, from categories to cosmology and from modules to Minkowski spacetime. The authors explain both aspects from scratch. A brief synopsis of the book goes as follows:

Chapters 1–3 constitute the basic foundation of the book. Chapter 1 describes the basic theory of algebras equipped with differential structure expressed as exterior algebras of differential forms, introducing the notion of a quantum metric and, in the case of an *inner* calculus, an induced quantum Laplacian, as elementary layers of the theory that depend only on the differential structure. Chapter 2 is a highly condensed introduction to Hopf algebras or *quantum groups* and their representations as monoidal or braided monoidal categories, developing the theory of differential structures and exterior algebras on Hopf algebras, including a braided antisymetric algebra approach to the Woronowitcz construction, and quantum/braided Lie algebras on quantum groups. Chapter 3 addresses the basic notions of a vector bundle over an algebra and of a connection on such a vector bundle over a differential algebra, covering elements of cyclic cohomology and K-theory, including the celebrated Chern-Connes pairing between the two.

Chapters 4–7, together with Chapter 1–3, form a fuller course for mathematicians. Chapter 4 proceeds to harder results about the curvature of connections such as the Bianchi identities and characteristic classes. The chapter also includes

- A study of the category of modules equipped with flat connections, which could be seen as playing the role of sheaves over the algebra [*E. J. Beggs* and *T. Brzeziński*, Acta Math. 195, No. 2, 155–196 (2005; Zbl 1130.46044)] (§4.3).
- Various constructions of sheaves and cohomology are given. To transfer between different ways of studying algebras in noncommutative algebraic geometry, a central role is taken by categories of quasicoherent sheaves [*J. T. Stafford* and *M. Van den Bergh*, Bull. Am. Math. Soc., New Ser. 38, No. 2, 171–216 (2001; Zbl 1042.16016)]. Group cohomology is described in [*K. S. Brown*, Cohomology of groups. New York-Heidelberg-Berlin: Springer (1982; Zbl 0584.20036)], and a key part of extending this cohomology to noncommutative geometry (§4.4.2) is extending this cohomology to Hopf algebras [*E. J. Beggs* and *T. Brzeziński*, Int. J. Geom. Methods Mod. Phys. 1, No. 1–2, 33–48 (2004; Zbl 1087.16024)]. The noncommutative van Est spectral sequence itself can be seen

as a unital coalgebra version of the standard cohomology of augmented algebras.

- Some applications of spectral sequences are given, including the Leray-Serre spectral sequence of a fibration for a differential definition of noncommutative fibration.
- The chapter deals with positive maps and C*-modules, extending the idea of bimodules as generalized morphisms between algebras to a differential setting in use of bimodule connections on B-A bimodules for different algebras A, B [*A. Connes* and *N. Higson*, C. R. Acad. Sci., Paris, Sér. I 311, No. 2, 101–106 (1990; Zbl 0717.46062)].

Chapter 5 addresses quantum principal bundles with the fiber being a Hopf algebra or quantum group [*T. Brzeziński* and *S. Majid*, Commun. Math. Phys. 157, No. 3, 591–638 (1993; Zbl 0817.58003); *T. Brzeziński* and *S. Majid*, Commun. Math. Phys. 167, No. 1, 235 (1995; Zbl 0823.58006); *S. Majid*, Banach Cent. Publ. 40, 335–349 (1997; Zbl 0884.58014); Phys. Lett. B. 298, 339–343 (1993)], which is a.k.a. a Hopf-Galois extension in the case of universal calculus. The chapter includes the following topics:

- The link with Galois theory is expounded, and the general theory of associated bundles and induced bimodule connections on them is provided when the principal bundle has a connection form. The link between classical Galois theory and Hopf-Galois theory has a long history [*S. U. Chase* and *M. E. Sweedler*, Hopf algebras and Galois theory. Berlin-Heidelberg-New York: Springer (1969; Zbl 0197.01403); *H. F. Kreimer* and *M. Takeuchi*, Indiana Univ. Math. J. 30, 675–692 (1981; Zbl 0451.16005)].
- The interaction of quantum principal bundle theory and bimodule connections occupying much of the chapter is new.
- Differential fibrations are investigated generally.
- The last section of the chapter (§5.6) is an application to quantum framed spaces, that is to say, differential algebras appearing as the base of a quantum principal bundle and data such that the space Ω^1 of differential 1-forms is an associated bundle [*S. Majid*, J. Geom. Phys. 30, No. 2, 113–146 (1999; Zbl 0940.58004); Commun. Math. Phys. 225, No. 1, 131–170 (2002; Zbl 0999.58004); Commun. Math. Phys. 256, No. 2, 255–285 (2005; Zbl 1075.58004)].

Chapter 6, consisting of five sections, develops the theory of vector fields and the algebra of differential operators $\mathcal{D}_A$ associated to an algebra with differential calculus, being a braided-commutative algebra in the center of the monoidal category of bimodules with bimodule connections. §6.2 (higher order differential operators) and §6.3 ($T\mathfrak{X}.$ as an algebra in $\mathcal{Z}({}_A\mathcal{E}_A)$), based largely on [*E. J. Beggs* and *T. Brzeziński*, J. Pure Appl. Algebra 218, No. 1, 1–17 (2014; Zbl 1279.32008)], addresses $T\mathfrak{X}.$, but Proposition 6.15 that

$${}_A\mathcal{E} \cong {}_{T\mathfrak{X}.}\mathcal{M}$$

is completely new.

Chapter 7, consisting of four sections, introduces complex structures after classical complex manifold theory, involving a bigrading of the exterior algebra to give a double complex and allowing the definition of holomorphic modules along with implications for cohomology theories. The authors discuss

- The integrability condition (§7.1) is a straightforwards generalization of the differential form version of the classical Newlander-Nirebberg integrability condition [*A. Newlander* and *L. Nirenberg*, Ann. Math. (2) 65, 391–404 (1957; Zbl 0079.16102)]. §7.2 includes aspects of the Koszul-Malgrange theorem.
- An early version of the classical Borel-Weil-Bott theorem [*M. Demazure*, Invent. Math. 33, 271–272 (1976; Zbl 0383.14017); *R. Bott*, Ann. Math. (2) 66, 203–248 (1957; Zbl 0094.35701)] for $\mathbb{C}_q[S^2]$ in terms of holomorphic sections was presented in [loc. cit., Zbl 1075.58004], while the authors' exposition of the cohomological version (§7.4.1) is completely new.
- A Borel-Weil-Bott theorem for higher-dimensional quantum Grassmannians can be seen in ["A Borel-Weil Theorem for the Quantum Grassmannians", Preprint, `arXiv:1611.07969`]. While the authors focus on the q-sphere as a q-deformed $\mathbb{CP}^1$ in the book, the same approach works for $\mathbb{CP}^{n-1}$ as a quantum homogeneous space for $\mathbb{C}_q[SU_n]$, the quantum principal bundle for which was given in [*U. Meyer*, Lett. Math. Phys. 35, No. 2, 91–97 (1995; Zbl 0847.17014)] with faithful flatness of $\mathbb{C}_q[SU_n]$ as a module over $\mathbb{C}_q[\mathbb{CP}^{n-1}]$ established in [*E. F. Müller* and *H. J. Schneider*, Isr. J. Math. 111, 157–190 (1999; Zbl 1001.17015)]. The full picture was gained in [*R. Ó Buachalla*, J. Geom. Phys. 99, 154–173 (2016; Zbl 1330.81130); Commun. Math. Phys. 316, No. 2, 345–373 (2012; Zbl

1269.81066)] as a calculus on $\mathbb{C}_q[SU_n]$ which restricts to [*I. Heckenberger* and *S. Kolb*, Proc. Lond. Math. Soc. (3) 89, No. 2, 457–484 (2004; Zbl 1056.58006)] on quantum complex projective space in just the same way as the Woronowicz's 3D calculus $\mathbb{C}_q[SU_n]$ [*S. L. Woronowicz*, Commun. Math. Phys. 122, No. 1, 125–170 (1989; Zbl 0751.58042)] restricts to a calculus on $\mathbb{C}_q[S^2]$.

Chapters 8 and 9, together with Chapter 1 form a short course for theoretical physicists. Chapter 8 brings together previously encountered notions of Riemannian and other structures from Chapters 3, 4 and 5 into a self-contained account of noncommutative Riemannian geoemtry over an algebra equipped with differential structure and choice of metric. More specifically,

- Finding an associated torsion free and metric compatible bimodule connection on Ω^1 is a well-posed nonlinear problem and the chapter shows how it can be solved directly in a variety of models.
- The chapter includes a section on Connes'spectral triples (§8.5) and it is shown how they can sometimes arise in a weakened form in the authors' approach as a Dirac operator built along geometric lines from a connection and a Clifford structure.
- The chapter includes a wave-operator approach to quantum Riemannian geoemtry, short-cutting the layer-by-layer treatment within the formalism of the quantum Laplacian as a partial derivative of an extended calculus.
- The chapter includes a slightly different theory of hermitian-metric compatible connections and Chern connections in noncommutaive geoemtry.

Chapter 9 discusses some applications of quantum Riemannian geometry to quantum spacetime and potential physical effects. The basic idea is that quantum spacetime, due to quantum gravity effects, is better described as a noncommutative or quantum geometry than a classical one.

- There are many points of view on the bicrosproduct model spacetime and §9.2 describes the original one as a quantization of flat space.
- §9.3 is concerned with the spherically symmetric case containing the black-hole [*S. Majid*, Commun. Math. Phys. 310, No. 3, 569–609 (2012; Zbl 1241.53060)].
- §9.4 is taken from [*E. J. Beggs* and *S. Majid*, Classical Quantum Gravity 31, No. 3, Article ID 035020, 39 p. (2014; Zbl 1290.83014)], presenting the first nontrivially curved quantum Riemannian geometry model that is to be fully solved and contains a bit of physics (a strong gravitational source or an expanding cosmology).
- §9.5 on the quantum Bertotti-Robinson model including AdS_2 and dS_2 is borrowed from [Cosmological constant from quantum spacetime, Shahn Majid and Wen-Qing Tao, Phys. Rev. D 91, 124028 – Published 9 June 2015].
- §9.6, taken from [*E. J. Beggs* and *S. Majid*, J. Geom. Phys. 114, 450–491 (2017; Zbl 1358.81129)], is the first attempt to systematically semiclassicalize noncommutative Riemannian geometry within the bimodule approach. §9.6.3, after [Nonassociative Riemannian geometry by twisting, Edwin Beggs1 and Shahn Majid, Journal of Physics: Conference Series, Volume 254, Quantum Groups, Quantum Foundations, and Quantum Information: a Festschrift for Tony Sudbery 29–30 September 2008, York, UK], is concerned with twisting to construct and control quasi-associative noncommutative geometry.

Some topics of great significance are merely touched or completely neglected in the book. Some of them are

- The authors briefly treat the semiclassical behavior within deformation theory or *Poisson-Riemannian geometry*, a paradigm including first-order quantum gravity effects and bearing the same relation to quantum gravity as does classical mechanics to quantum mechanics.
- The authors briefly treat the construction of examples by *functorial twisting*, which is particularly interesting in the cochain case where the data on the symmetry quantum group is not a cocycle and the exterior algebra becomes nonassociative.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

53-02 Research exposition (monographs, survey articles) pertaining to differential geometry
53Cxx Global differential geometry
81-02 Research exposition (monographs, survey articles) pertaining to quantum theory
81Rxx Groups and algebras in quantum theory

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