Articles

Dynamical stability in strategic communication with the information structure and perturbations

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Abstract

We investigate dynamical stability for strategic communication with the information structure and perturbations under the replicator dynamics. To extend the theoretical framework proposed by Green and Stokey (2007), we study dynamical stability of all equilibria with the information structure which they introduced. We show that the rest points of one kind of partition equilibrium and a determinate action equilibrium can be stable under the replicator dynamics in the case where there are two states, two actions, and two observations. Moreover, we reveal the effects of the information structure and perturbations on the dynamical behavior. Without the information structure, dynamical stability depends on fewer elements of utility functions and beliefs of an agent and a principal than with the information structure. Perturbations of the repilicator dynamics can stabilize complete communication that has an unstable rest point under the replicator dynamics.

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1 Introduction

We consider the dynamical behavior for strategic communication with the inforfmation structure and perturbations under the replicator dynamics. In the strategic communication which is studied in economics (Crawford and Sobel, 1982, Green and Stokey, 2007), all members communicate through the strategic use of signals. We study effects of the information structure and perturbations on the dynamical behavior under the replicator dynamics.

The starting point for analysis is the model of information transmission as studied by Green and Stokey (2007). In this game, after a state of nature occurs, an agent receives an observation related to the state through an information structure and sends it to a principal. The principal takes the decision. Each utility of the agent and the principal depends on the state and the action.

In this game, there are sets of equilibria. Following Green and Stokey (2007), we focus on three types of equilibria: a partition equilibrium, a determinate action equilibrium, and a random action equilibrium. We study the dynamical behavior of these equilibria in the case where there are two states, two actions, and two observations. In addition, we suppose that beliefs for an agent and a principal are identical.

We first study rest points of these equilibria and dynamical stability of these rest points with the information structure under the replicator dynamics. Sequentially, we study effects of the information structure on the dynamical behavior under the replicator dynamics.

Next, we study the dynamical behavior of complete communication (one kind of partition equilibrium) under the replicator dynamics with perturbations which is called the selection–mutation dynamics (Hofbauer, 1985). The dynamical behavior of strategic communication with common interest under the selection–mutation dynamics is studied by Hofbauer and Huttegger (2007, 2015) and Uchida and Fukuzumi (2019). Two of them show that perturbations of the replicator dynamics can stabilize the dynamical behavior (Hofbauer and Huttegger, 2007; Uchida and Fukuzumi, 2019).

Our work makes three important contributions:

- We show that a partition equilibrium has a rest point under the replicator dynamics, and that a determinate action equilibrium and a random action equilibrium can be the rest point under the replicator dynamics. Moreover, we also show that rest points of one kind of partition equilibrium and a determinate action equilibrium can be stable under the replicator dynamics.

- We show that without the information structure, the dynamical stability of the strategic communication depends on fewer elements of utility functions and beliefs of an agent and a principal than with the information structure.

- We show that a rest point close to complete communication with the information structure that has an unstable rest point under the replicator dynamics can be asymptotically stable under the selection–mutation dynamics.

The remainder of this paper is organized as follows. Section 2 provides the formal model of strategic communication with the information structure. Section 3 introduces types of equilibria and Section 4 introduces the dynamics. Section 5 studies the stability of these equilibria under the replicator dynamics. Section 6 studies the stability of complete communication under the selection–mutation dynamics. Section 7 concludes.

2 Model

Our decision problems consist of two players; one is an agent and the other is a principle. There are m states of the world by the set $\Theta = \{\theta_1, ..., \theta_M\}$, and N possible observations y_n by the set $Y = \{y_1, ..., y_N\}$. An observation is statistically related to the true state in Θ. The statistical relationship between states and observations is called the information structure. It is represented by an $M \times N$ Markov matrix $\Lambda = [\lambda_{mn}]$. λ_{mn} is the probability tha y_n is observed if the true state is θ_m. There are k actions by the set $A = \{a_1, ..., a_K\}$.

An agent receives an observation and sends it to a principal. However, the agent may not send the same information as he observed. A principal receives the information from the agent and chooses an action a_k from A.

The von Neumann-Morgenstern utility levels of two players depend upon the action and the state of nature. These utilities are represented by $K \times M$ matrices $U^A_{km} = [u^a_{km}]$ and $U^P_{km} = [u^p_{km}]$ for the principal and the agent. We suppose that all elements of $U^A_{km} = [u^a_{km}]$ and $U^P_{km} = [u^p_{km}]$ are positive. U^A_{km} and U^P_{km} for the principal and the agent are realized if θ_m occurs and a_k is chosen by the principal.

The agent's strategies are represented by an $N \times N$ Markov matrix

$$R \in R^{\triangle}_{N\times N} = \{R \in \mathcal{R}^{N\times N}_{+} : \sum_{j=1}^{N} r_{ij} = 1, \forall i \in N\},$$

where $r_{nn'}$ is the probability that $y_{n'}$ was sent given that the actual observation is y_n.

The principal chooses the action $a_k \in A$ given that the information y'_n was sent by him. The principal's strategy is represented by an $N \times K$ Markov matrix

$$Z \in R^{\triangle}_{N\times K} = \{R \in \mathcal{R}^{N\times K}_{+} : \sum_{i=1}^{K} z_{ji} = 1, \forall j \in N\},$$

where $z_{n'k}$ is the probability that a_k is chosen given that $y_{n'}$ was sent.

We denote different beliefs for the agent and the principal by $\pi = (\pi_1, ..., \pi_M) \in \Delta^M$ and $\pi' = (\pi'_1, ..., \pi'_M) \in \Delta^M$ where Δ^M is the set of all M-dimensional probability vectors.

If the strategy choices are Z and R, the expected utilities for the agent and the principal are respectively

$$tr\Pi\Lambda RZU^A$$

and

$$tr\Pi'\Lambda RZU^P$$

where Π and Π' denote the square matrices with the vectors π and π' on the diagonal and zeros elsewhere. [1]

Our game $\Gamma_{m,n} = \{R^{\triangle}_{N\times N} \times Z^{\triangle}_{N\times K}, trU\Pi\Lambda RZ, trU'\Pi'\Lambda RZ\}$ is described.

We study the Nash equilibria of this game. Let $B(Z) \in R^{\triangle}_{N\times N}$ and $B(R) \in Z^{\triangle}_{N\times K}$ denote the best-response correspondence of R and Z respectively.

Lemma 1. A pair $(R, Z) \in R^{\triangle}$ is a **Nash strategy** of $\Gamma_{N,K}$ if and only if $R \in B(Z)$ and $Z \in B(R)$.

3 Types of equilibria

$\Gamma_{N,K}$ has a large set of equilibria. Following Green and Stokey (2007), we provide basic classifications of equilibria. First, we provide characteristics of the information structure.

Definition 1 We say that an $M \times N'$ information structure Λ' is a partition of Λ if $\Lambda' = \Lambda PDP'$, where P and P' are permutation matrices and D is an $N \times N'$ block diagonal Markov matrix in which each block has rank one.

When $\Lambda' = \Lambda PDP'$, there is a partition of the information space Y. If

[1] The agent's expected utility is represented by

$$EU^A = \sum_m \pi_m \sum_n \lambda_{mn} \sum_{n'} r_{nn'} \sum_k z_{n'k} u^a_{km}.$$

The principal's expected utility is represented by

$$EU^P = \sum_m \pi'_m \sum_n \lambda_{mn} \sum_{n'} r_{nn'} \sum_k z_{n'k} u^p_{km}.$$

information value y_k occurs under Λ', the partition element containing y_k is reported through Λ'.

An equilibrium in this information structure, $\Lambda' = \Lambda PDP'$, is called *a partition equilibrium* (Green and Stokey, 2007). In this equilibrium, ΛR is a partition of Λ. In short, the observation that an agent received is sent as itself or as partitioned information.

Definition 2 (Green and Stokey, 2007) We say that an equilibrium pair (R, Z) is **a partition equilibrium** if $\Lambda' = \Lambda R$ is a partition of Λ. One equilibrium of this type is the pair of strategies $R = I, Z = I$ and $RZ = I$. Another equilibrium is the pair of strategies in which Z has at least one zero-column.

We consider the case in which there are two states, two actions, and two observations. A partition equilibrium is represented by two forms

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{pmatrix}, Z_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \alpha \in [0, 1].$$

In addition to a partition equilibrium, there are two types of non–partition equilibria in which a principal uses a pure or mixed strategy.

Definition 3 (Green and Stokey, 2007) We say that an equilibrium pair (R, Z) is **a determinate action equilibrium** if $\Lambda' = \Lambda R$ is not a partition of Λ, and each row of Z receiving positive weight under R has only a single positive element.

In the case where there are two states, two actions, and two observations, a determinate action equilibrium is represented as follows:

$$R_3 = \begin{pmatrix} 1-\alpha & \alpha \\ 0 & 1 \end{pmatrix}, Z_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 < \alpha \leq \frac{1}{2}.$$

Definition 4 (Green and Stokey, 2007) We say that an equilibrium pair (R, Z) is **a random action equilibrium** if $\Lambda' = \Lambda R$ is not a partition of Λ, and some row of Z receiving positive weight under R has two or more nonzero entries.

In the case where there are two states, two actions, and two observations, a random action equilibrium is represented by the form:

$$R_4 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, Z_4 = \begin{pmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{pmatrix}, \alpha \in (0, 1).$$

4 Dynamics

We now consider the replicator dynamics and the selection–mutation dynamics on the behavioral strategies, as per Hofbauer and Hutteger (2015). In an extensive form of this game, a behavioral strategy is represented by a probability measure over strategies of an agent and a principal.

We define an $(n-1)$-dimensional behavioral strategy simplex of an agent when the agent receives an observation $i \in N$, defined by S_i, as

$$S_i = \{(r_{i1}, r_{i2}, \ldots, r_{in}) | \sum_{j=1}^{n} r_{ij} = 1, r_{ij} \geq 0 \text{ for each } j \in N\}.$$

Similarly, we define a $(k-1)$-dimensional behavioral strategy simplex of an principal when the principal receives the information $j \in N$, defined by S_j, as

$$S_j = \{(z_{j1}, z_{j2}, \dots, z_{jk}) | \sum_{l=1}^{n} z_{jl} = 1, z_{jl} \geq 0 \text{ for each } j \in N\}.$$

The space of behavioral strategies is defined by $S = \Pi_{i \in N} S_i \times \Pi_{j \in K} S_j$.

Our dynamic selection process is described by a dynamical system of differential equations defined for all points in S. In this paper, we consider the case in which there are two states, two actions, and two observations. The dynamical system is formulated as the following 8 differential equations:

$$\begin{aligned}
\dot{r}_{11} &= r_{11}(\pi_1\lambda_{11}z_{11}u^a_{11} + \pi_1\lambda_{11}z_{12}u^a_{21} + \pi_2\lambda_{21}z_{11}u^a_{12} + \pi_2\lambda_{21}z_{12}u^a_{22} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{11}u^a_{11} - \pi_1\lambda_{11}r_{11}z_{12}u^a_{21} - \pi_2\lambda_{21}r_{11}z_{11}u^a_{12} - \pi_2\lambda_{21}r_{11}z_{12}u^a_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{21}u^a_{11} - \pi_1\lambda_{11}r_{12}z_{22}u^a_{21} - \pi_2\lambda_{21}r_{12}z_{21}u^a_{12} - \pi_2\lambda_{21}r_{12}z_{22}u^a_{22}) + \varepsilon(1 - 2r_{11}), \\
\dot{r}_{12} &= r_{12}(\pi_1\lambda_{11}z_{21}u^a_{11} + \pi_1\lambda_{11}z_{22}u^a_{21} + \pi_2\lambda_{21}z_{21}u^a_{12} + \pi_2\lambda_{21}z_{22}u^a_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{21}u^a_{11} - \pi_1\lambda_{11}r_{12}z_{22}u^a_{21} - \pi_2\lambda_{21}r_{12}z_{21}u^a_{12} - \pi_2\lambda_{21}r_{12}z_{22}u^a_{22} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{11}u^a_{11} - \pi_1\lambda_{11}r_{11}z_{12}u^a_{21} - \pi_2\lambda_{21}r_{11}z_{11}u^a_{12} - \pi_2\lambda_{21}r_{11}z_{12}u^a_{22}) + \varepsilon(1 - 2r_{12}), \\
\dot{r}_{21} &= r_{21}(\pi_1\lambda_{12}z_{11}u^a_{11} + \pi_1\lambda_{12}z_{12}u^a_{21} + \pi_2\lambda_{22}z_{11}u^a_{12} + \pi_2\lambda_{22}z_{12}u^a_{22} \\
&\quad -\pi_1\lambda_{12}r_{21}z_{11}u^a_{11} - \pi_1\lambda_{12}r_{21}z_{12}u^a_{21} - \pi_2\lambda_{22}r_{21}z_{11}u^a_{12} - \pi_2\lambda_{22}r_{21}z_{12}u^a_{22} \\
&\quad -\pi_1\lambda_{12}r_{22}z_{21}u^a_{11} - \pi_1\lambda_{12}r_{22}z_{22}u^a_{21} - \pi_2\lambda_{22}r_{22}z_{21}u^a_{12} - \pi_2\lambda_{22}r_{22}z_{22}u^a_{22}) + \varepsilon(1 - 2r_{21}), \\
\dot{r}_{22} &= r_{22}(\pi_1\lambda_{12}z_{21}u^a_{11} + \pi_1\lambda_{12}z_{22}u^a_{21} + \pi_2\lambda_{22}z_{21}u^a_{12} + \pi_2\lambda_{22}z_{22}u^a_{22} \\
&\quad -\pi_1\lambda_{12}r_{22}z_{21}u^a_{11} - \pi_1\lambda_{12}r_{22}z_{22}u^a_{21} - \pi_2\lambda_{22}r_{22}z_{21}u^a_{12} - \pi_2\lambda_{22}r_{22}z_{22}u^a_{22} \\
&\quad -\pi_1\lambda_{12}r_{21}z_{11}u^a_{11} - \pi_1\lambda_{12}r_{21}z_{12}u^a_{21} - \pi_2\lambda_{22}r_{21}z_{11}u^a_{12} - \pi_2\lambda_{22}r_{21}z_{12}u^a_{22}) + \varepsilon(1 - 2r_{22}), \\
\dot{z}_{11} &= z_{11}(\pi_1\lambda_{11}r_{11}u^p_{11} + \pi_1\lambda_{12}r_{21}u^p_{11} + \pi_2\lambda_{21}r_{11}u^p_{12} + \pi_2\lambda_{22}r_{21}u^p_{12} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{11}u^p_{11} - \pi_1\lambda_{12}r_{21}z_{11}u^p_{11} - \pi_2\lambda_{21}r_{11}z_{11}u^p_{12} - \pi_2\lambda_{22}r_{21}z_{11}u^p_{12} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{12}u^p_{21} - \pi_1\lambda_{12}r_{21}z_{12}u^p_{21} - \pi_2\lambda_{21}r_{11}z_{12}u^p_{22} - \pi_2\lambda_{22}r_{21}z_{12}u^p_{22}) + \delta(1 - 2z_{11}), \\
\dot{z}_{12} &= z_{12}(\pi_1\lambda_{11}r_{11}u^p_{21} + \pi_1\lambda_{12}r_{21}u^p_{21} + \pi_2\lambda_{21}r_{11}u^p_{22} + \pi_2\lambda_{22}r_{21}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{12}u^p_{21} - \pi_1\lambda_{12}r_{21}z_{12}u^p_{21} - \pi_2\lambda_{21}r_{11}z_{12}u^p_{22} - \pi_2\lambda_{22}r_{21}z_{12}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{11}u^p_{11} - \pi_1\lambda_{12}r_{21}z_{11}u^p_{11} - \pi_2\lambda_{21}r_{11}z_{11}u^p_{12} - \pi_2\lambda_{22}r_{21}z_{11}u^p_{12}) + \delta(1 - 2z_{12}),
\end{aligned}$$

$$
\begin{aligned}
\dot{z}_{21} &= z_{21}(\pi_1\lambda_{11}r_{12}u^p_{11} + \pi_1\lambda_{12}r_{22}u^p_{11} + \pi_2\lambda_{21}r_{12}u^p_{12} + \pi_2\lambda_{22}r_{22}u^p_{12} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{22}u^p_{21} - \pi_1\lambda_{12}r_{22}z_{22}u^p_{21} - \pi_2\lambda_{21}r_{12}z_{22}u^p_{22} - \pi_2\lambda_{22}r_{22}z_{22}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{21}u^p_{11} - \pi_1\lambda_{12}r_{22}z_{21}u^p_{11} - \pi_2\lambda_{21}r_{12}z_{21}u^p_{12} - \pi_2\lambda_{22}r_{22}z_{21}u^p_{12}) + \delta(1-2z_{21}), \\
\dot{z}_{22} &= z_{22}(\pi_1\lambda_{11}r_{12}u^p_{21} + \pi_1\lambda_{12}r_{22}u^p_{21} + \pi_2\lambda_{21}r_{12}u^p_{22} + \pi_2\lambda_{22}r_{22}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{22}u^p_{21} - \pi_1\lambda_{12}r_{22}z_{22}u^p_{21} - \pi_2\lambda_{21}r_{12}z_{22}u^p_{22} - \pi_2\lambda_{22}r_{22}z_{22}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{21}u^p_{11} - \pi_1\lambda_{12}r_{22}z_{21}u^p_{11} - \pi_2\lambda_{21}r_{12}z_{21}u^p_{12} - \pi_2\lambda_{22}r_{22}z_{21}u^p_{12}) + \delta(1-2z_{22}).
\end{aligned}
$$

where ε and δ are small, uniform mutation rates.

We denote this system by $\dot{S} = \Phi(S)$. This dynamical system is called the **selection–mutation dynamics** (Hofbauer, 1985). If $\varepsilon = \delta = 0$, the selection–mutation dynamics coincides with the **replicator dynamics**.

5 Dynamical stability under the replicator dynamics

In this section, we study dynamical stability under the replicator dynamics in the case where there are two states, two actions, and two observations. In addition, we suppose that beliefs for an agent and a principal are identical. In the following results, we first check rest points of three types of equilibria. After that, we study the dynamical stability of these rest points.

Theorem 5.1. Let (R_1, Z_1) and (R_2, Z_2) be partition equilibria of Definition 2 in the case where there are two states, two actions, and two observations. Then, the partition equilibria (R_1, Z_1) and (R_2, Z_2) have rest points under the replicator dynamics. The rest point (R_1, Z_1) is structurally stable under the replicator dynamics when

$$\begin{aligned}
&\pi_1\lambda_{11}u^a_{21} - \pi_1\lambda_{11}u^a_{11} + \pi_2\lambda_{21}u^a_{22} - \pi_2\lambda_{21}u^a_{12} < 0,\\
&\pi_1\lambda_{12}u^a_{11} - \pi_1\lambda_{12}u^a_{21} + \pi_2\lambda_{22}u^a_{12} - \pi_2\lambda_{22}u^a_{22} < 0,\\
&\pi_1\lambda_{11}u^p_{21} - \pi_1\lambda_{11}u^p_{11} + \pi_2\lambda_{21}u^p_{22} - \pi_2\lambda_{21}u^p_{12} < 0,\\
&\pi_1\lambda_{12}u^p_{11} - \pi_1\lambda_{12}u^p_{21} + \pi_2\lambda_{22}u^p_{12} - \pi_2\lambda_{22}u^p_{22} < 0.
\end{aligned}$$

On the other hand, the rest point (R_2, Z_2) is structurally unstable under the replicator dynamics.

Theorem 5.2. Let (R, Z) be a determinate action equilibrium in the case where there are two states, two actions, and two observations. Then, the determinate action equilibrium has a rest point under the replicator dynamics when

$$\pi_1\lambda_{11}u^a_{11} - \pi_1\lambda_{11}u^a_{21} + \pi_2\lambda_{21}u^a_{12} - \pi_2\lambda_{21}u^a_{22} = 0.$$

The rest point of (R, Z) is structurally stable under the replicator dynamics when

$$\begin{aligned}
&(2\alpha-1)\pi_1\lambda_{11}u^a_{11} + (2\alpha-1)\pi_2\lambda_{21}u^a_{12} - \alpha\pi_1\lambda_{11}u^a_{21} - \alpha\pi_2\lambda_{21}u^a_{22}) < 0,\\
&(1-2\alpha)\pi_1\lambda_{11}u^a_{21} + (1-2\alpha)\pi_2\lambda_{21}u^a_{22} - (1-\alpha)\pi_1\lambda_{11}u^a_{11} - (1-\alpha)\pi_2\lambda_{21}u^a_{12}) < 0,\\
&\pi_1\lambda_{12}u^a_{11} - \pi_1\lambda_{12}u^a_{21} + \pi_2\lambda_{22}u^a_{12} - \pi_2\lambda_{22}u^a_{22} < 0,\\
&\pi_1\lambda_{11}(u^p_{21} - u^p_{11}) + \pi_2\lambda_{21}(u^p_{22} - u^p_{12}) < 0,\\
&\alpha\pi_1\lambda_{11}(u^p_{11} - u^p_{21}) + \pi_1\lambda_{12}(u^p_{11} - u^p_{21}) + \alpha\pi_2\lambda_{21}(u^p_{12} - u^p_{22}) - \pi_2\lambda_{22}(u^p_{12} - u^p_{22}) < 0.
\end{aligned}$$

Theorem 5.3. Let (R, Z) be a random action equilibrium in the case where there are two states, two actions, and two observations. Then, the random action equilibrium has a rest point under the replicator dynamics when

$$
\begin{aligned}
&\pi_1\lambda_{11}u^a_{11} - \pi_1\lambda_{11}u^a_{21} + \pi_2\lambda_{21}u^a_{12} - \pi_2\lambda_{21}u^a_{22} = 0,\\
&\tfrac{1}{2}\pi_1\lambda_{11}u^p_{21} - \tfrac{1}{2}\pi_1\lambda_{11}u^p_{11} + \pi_1\lambda_{12}u^p_{21} - \pi_1\lambda_{12}u^p_{11}\\
&+\tfrac{1}{2}\pi_2\lambda_{21}u^p_{22} - \tfrac{1}{2}\pi_2\lambda_{21}u^p_{12} + \pi_2\lambda_{22}u^p_{22} - \pi_2\lambda_{22}u^p_{12} = 0.
\end{aligned}
$$

The rest point of (R, Z) is structurally unstable under the replicator dynamics.

Green and Stokey (2007) studied the welfare of an agent and a principal when the information structure changes. We also study the role of the information structure from the point of view of dynamical stability.

Theorem 5.4. Suppose that there is no information structure. Then, the partition equilibrium (R_1, Z_1) of Definition 2 has a rest point under the replicator dynamics. The rest point is structurally stable under the replicator dynamics when $u^a_{21} - u^a_{11} < 0, u^a_{12} - u^a_{22} < 0, u^p_{21} - u^p_{11} < 0$, and $u^p_{12} - u^p_{22} < 0$.

Without the information structure, dynamical stability of the partition equilibrium (R_1, Z_1) depends only on utility functions as opposed to with the information structure.

Theorem 5.5. Suppose that there is no information structure. Then, a determinate action equilibrium (R_3, Z_3) has a rest point under the replicator dynamics when

$$u^a_{11} = u^a_{21}.$$

The rest point is structurally stable under the replicator dynamics when

$$
\begin{aligned}
&-\alpha u^a_{21} + (1-2\alpha)u^a_{11} < 0,\\
&(1-2\alpha)u^a_{21} - (1-\alpha)u^a_{11} < 0,\\
&u^a_{12} - u^a_{22} < 0,\\
&u^p_{21} - u^p_{11} < 0,\\
&\pi_1\alpha u^p_{11} + \pi_2 u^p_{12} - \pi_1\alpha u^p_{21} - \pi_2 u^p_{22} < 0.
\end{aligned}
$$

Without the information structure, the condition of a rest point at a determinate action equilibrium is simply $u^a_{11} = u^a_{21}$ regardless of $u^a_{12}, u^a_{22}, \pi_1$ and π_2 as opposed to with the information structure. Moreover, the condition of stability at a determinate action equilibrium depends on fewer elements of utility functions and beliefs of an agent and a principal than with the information structure.

We can check that a partition equilibrium (R_2, Z_2) of Definition 2 and a random action equilibrium without the information structure are structurally unstable. [2]

6 Dynamical stability under the selection–mutation dynamics

Next, we study dynamical stability under the selection–mutation dynamics in the case where there are two states, two actions, and two observations. In addition, we suppose that beliefs for an agent and a principal are identical. First, we study the rest point close to (R_1, Z_1) under the selection–mutation dynamics.

Theorem 6.1. Consider a partition equilibrium (R_1, Z_1) in the case where there are two states, two actions, and two observations. For each pair of the mutation

[2]We omit the proof of these cases due to lack of space.

rates (ε, δ), there is a neighborhood of the partition equilibrium (R_1, Z_1) that contains a unique rest point $(R_1^*(\varepsilon, \delta), Z_1^*(\varepsilon, \delta))$ when

$$\begin{aligned}
&\pi_1\lambda_{11}u^a_{21} + \pi_2\lambda_{21}u^a_{22} - \pi_1\lambda_{11}u^a_{11} - \pi_2\lambda_{21}u^a_{12} \neq 0,\\
&\pi_1\lambda_{12}u^a_{11} + \pi_2\lambda_{22}u^a_{12} - \pi_1\lambda_{12}u^a_{21} - \pi_2\lambda_{22}u^a_{22} \neq 0,\\
&\pi_1\lambda_{11}u^p_{21} + \pi_2\lambda_{21}u^p_{22} - \pi_1\lambda_{11}u^p_{11} - \pi_2\lambda_{21}u^p_{12} \neq 0,\\
&\pi_1\lambda_{12}u^p_{11} + \pi_2\lambda_{22}u^p_{12} - \pi_1\lambda_{12}u^p_{21} - \pi_2\lambda_{22}u^p_{22} \neq 0.
\end{aligned}$$

We find a rest point close to a partition equilibrium (R_1, Z_1) under the selection–mutation dynamics. We can show the value of the rest point explicitly.

Corollary 1. The first–order approximated entries of the rest point $(R(\varepsilon, \delta), Z(\varepsilon, \delta)) \in S$ close to a partition equilibrium (R_1, Z_1) are explicitly given as follows:

$$R_1 = \begin{pmatrix} 1 - \frac{1}{\pi_1\lambda_{11}u^a_{11}+\pi_2\lambda_{21}u^a_{12}-\pi_1\lambda_{11}u^a_{21}-\pi_2\lambda_{21}u^a_{22}}\varepsilon & \frac{1}{\pi_1\lambda_{11}u^a_{11}+\pi_2\lambda_{21}u^a_{12}-\pi_1\lambda_{11}u^a_{21}-\pi_2\lambda_{21}u^a_{22}}\varepsilon \\ \frac{1}{\pi_1\lambda_{12}u^a_{21}+\pi_2\lambda_{22}u^a_{22}-\pi_1\lambda_{12}u^a_{11}-\pi_2\lambda_{22}u^a_{12}}\varepsilon & 1 - \frac{1}{\pi_1\lambda_{12}u^a_{21}+\pi_2\lambda_{22}u^a_{22}-\pi_1\lambda_{12}u^a_{11}-\pi_2\lambda_{22}u^a_{12}}\varepsilon \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 1 - \frac{1}{\pi_1\lambda_{11}u^p_{11}+\pi_2\lambda_{21}u^p_{12}-\pi_1\lambda_{11}u^p_{21}-\pi_2\lambda_{21}u^p_{22}}\delta & \frac{1}{\pi_1\lambda_{11}u^p_{11}+\pi_2\lambda_{21}u^p_{12}-\pi_1\lambda_{11}u^p_{21}-\pi_2\lambda_{21}u^p_{22}}\delta \\ \frac{1}{\pi_1\lambda_{12}u^p_{21}+\pi_2\lambda_{22}u^p_{22}-\pi_1\lambda_{12}u^p_{11}-\pi_2\lambda_{22}u^p_{12}}\delta & 1 - \frac{1}{\pi_1\lambda_{12}u^p_{21}+\pi_2\lambda_{22}u^p_{22}-\pi_1\lambda_{12}u^p_{11}-\pi_2\lambda_{22}u^p_{12}}\delta. \end{pmatrix}.$$

We can study the stability of a rest point under the selection–mutation dynamics by using these values of Corollary 1.

Theorem 6.2. Consider a partition equilibrium (R_1, Z_1) that has an unstable rest point under the replicator dynamics in the case where there are two states, two actions, and two observations. Then, the rest point close to the corresponding partition equilibrium (R_1^*, Z_1^*) can be asymptotically stable under the selection–mutation dynamics.

Perturbations of the replicator dynamics can stabilize the dynamical behavior for the strategic communication of Green and Stokey (2007) type, following the dynamical behavior for the sender–receiver game of Lewis type (Hofbauer and Huttegger, 2007, Uchida and Fukuzumi, 2019). In this paper, we do not study stability of the other equilibrium under the selection–mutation dynamics because it is too difficult to solve the characteristic equation of the first-order approximated Jacobian matrix evaluated at the rest point.

7 Conclusion

In this paper, we study the rest points and dynamical stability for the strategic communication with the information structure and perturbations. With the information structure, the existence of rest points and dynamical stability for strategic communication depends on more elements of utility functions and more beliefs of the agent and the principal than without the information structure. On the other hand, perturbations of the replicator dynamics can stabilize the dynamical behavior of complete communication that has an unstable rest point under the replicator dynamics.

Appendix

Proof of Theorem 5.1

We consider the case in which there are two states, two actions, and two observations:

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, U^A = \begin{pmatrix} u^a_{11} & u^a_{12} \\ u^a_{21} & u^a_{22} \end{pmatrix}, U^P = \begin{pmatrix} u^p_{11} & u^p_{12} \\ u^p_{21} & u^p_{22} \end{pmatrix}, \pi^A = \pi^P = (\pi_1, \pi_2).$$

Partition equilibria are represented by two forms:

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{pmatrix}, Z_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Our dynamical system $S' = \Phi'(S)$ of the replicator dynamics consists of 8 differential equations:

$$\begin{aligned}
\dot{r}_{11} &= r_{11}(\pi_1\lambda_{11}z_{11}u^a_{11} + \pi_1\lambda_{11}z_{12}u^a_{21} + \pi_2\lambda_{21}z_{11}u^a_{12} + \pi_2\lambda_{21}z_{12}u^a_{22} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{11}u^a_{11} - \pi_1\lambda_{11}r_{11}z_{12}u^a_{21} - \pi_2\lambda_{21}r_{11}z_{11}u^a_{12} - \pi_2\lambda_{21}r_{11}z_{12}u^a_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{21}u^a_{11} - \pi_1\lambda_{11}r_{12}z_{22}u^a_{21} - \pi_2\lambda_{21}r_{12}z_{21}u^a_{12} - \pi_2\lambda_{21}r_{12}z_{22}u^a_{22}), \\
\dot{r}_{12} &= r_{12}(\pi_1\lambda_{11}z_{21}u^a_{11} + \pi_1\lambda_{11}z_{22}u^a_{21} + \pi_2\lambda_{21}z_{21}u^a_{12} + \pi_2\lambda_{21}z_{22}u^a_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{21}u^a_{11} - \pi_1\lambda_{11}r_{12}z_{22}u^a_{21} - \pi_2\lambda_{21}r_{12}z_{21}u^a_{12} - \pi_2\lambda_{21}r_{12}z_{22}u^a_{22} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{11}u^a_{11} - \pi_1\lambda_{11}r_{11}z_{12}u^a_{21} - \pi_2\lambda_{21}r_{11}z_{11}u^a_{12} - \pi_2\lambda_{21}r_{11}z_{12}u^a_{22}), \\
\dot{r}_{21} &= r_{21}(\pi_1\lambda_{12}z_{11}u^a_{11} + \pi_1\lambda_{12}z_{12}u^a_{21} + \pi_2\lambda_{22}z_{11}u^a_{12} + \pi_2\lambda_{22}z_{12}u^a_{22} \\
&\quad -\pi_1\lambda_{12}r_{21}z_{11}u^a_{11} - \pi_1\lambda_{12}r_{21}z_{12}u^a_{21} - \pi_2\lambda_{22}r_{21}z_{11}u^a_{12} - \pi_2\lambda_{22}r_{21}z_{12}u^a_{22} \\
&\quad -\pi_1\lambda_{12}r_{22}z_{21}u^a_{11} - \pi_1\lambda_{12}r_{22}z_{22}u^a_{21} - \pi_2\lambda_{22}r_{22}z_{21}u^a_{12} - \pi_2\lambda_{22}r_{22}z_{22}u^a_{22}), \\
\dot{r}_{22} &= r_{22}(\pi_1\lambda_{12}z_{21}u^a_{11} + \pi_1\lambda_{12}z_{22}u^a_{21} + \pi_2\lambda_{22}z_{21}u^a_{12} + \pi_2\lambda_{22}z_{22}u^a_{22} \\
&\quad -\pi_1\lambda_{12}r_{22}z_{21}u^a_{11} - \pi_1\lambda_{12}r_{22}z_{22}u^a_{21} - \pi_2\lambda_{22}r_{22}z_{21}u^a_{12} - \pi_2\lambda_{22}r_{22}z_{22}u^a_{22} \\
&\quad -\pi_1\lambda_{12}r_{21}z_{11}u^a_{11} - \pi_1\lambda_{12}r_{21}z_{12}u^a_{21} - \pi_2\lambda_{22}r_{21}z_{11}u^a_{12} - \pi_2\lambda_{22}r_{21}z_{12}u^a_{22}),
\end{aligned}$$

$$
\begin{aligned}
\dot{z}_{11} &= z_{11}(\pi_1\lambda_{11}r_{11}u^p_{11} + \pi_1\lambda_{12}r_{21}u^p_{11} + \pi_2\lambda_{21}r_{11}u^p_{12} + \pi_2\lambda_{22}r_{21}u^p_{12} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{11}u^p_{11} - \pi_1\lambda_{12}r_{21}z_{11}u^p_{11} - \pi_2\lambda_{21}r_{11}z_{11}u^p_{12} - \pi_2\lambda_{22}r_{21}z_{11}u^p_{12} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{12}u^p_{21} - \pi_1\lambda_{12}r_{21}z_{12}u^p_{21} - \pi_2\lambda_{21}r_{11}z_{12}u^p_{22} - \pi_2\lambda_{22}r_{21}z_{12}u^p_{22}), \\
\dot{z}_{12} &= z_{12}(\pi_1\lambda_{11}r_{11}u^p_{21} + \pi_1\lambda_{12}r_{21}u^p_{21} + \pi_2\lambda_{21}r_{11}u^p_{22} + \pi_2\lambda_{22}r_{21}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{12}u^p_{21} - \pi_1\lambda_{12}r_{21}z_{12}u^p_{21} - \pi_2\lambda_{21}r_{11}z_{12}u^p_{22} - \pi_2\lambda_{22}r_{21}z_{12}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{11}z_{11}u^p_{11} - \pi_1\lambda_{12}r_{21}z_{11}u^p_{11} - \pi_2\lambda_{21}r_{11}z_{11}u^p_{12} - \pi_2\lambda_{22}r_{21}z_{11}u^p_{12}), \\
\dot{z}_{21} &= z_{21}(\pi_1\lambda_{11}r_{12}u^p_{11} + \pi_1\lambda_{12}r_{22}u^p_{11} + \pi_2\lambda_{21}r_{12}u^p_{12} + \pi_2\lambda_{22}r_{22}u^p_{12} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{22}u^p_{21} - \pi_1\lambda_{12}r_{22}z_{22}u^p_{21} - \pi_2\lambda_{21}r_{12}z_{22}u^p_{22} - \pi_2\lambda_{22}r_{22}z_{22}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{21}u^p_{11} - \pi_1\lambda_{12}r_{22}z_{21}u^p_{11} - \pi_2\lambda_{21}r_{12}z_{21}u^p_{12} - \pi_2\lambda_{22}r_{22}z_{21}u^p_{12}), \\
\dot{z}_{22} &= z_{22}(\pi_1\lambda_{11}r_{12}u^p_{21} + \pi_1\lambda_{12}r_{22}u^p_{21} + \pi_2\lambda_{21}r_{12}u^p_{22} + \pi_2\lambda_{22}r_{22}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{22}u^p_{21} - \pi_1\lambda_{12}r_{22}z_{22}u^p_{21} - \pi_2\lambda_{21}r_{12}z_{22}u^p_{22} - \pi_2\lambda_{22}r_{22}z_{22}u^p_{22} \\
&\quad -\pi_1\lambda_{11}r_{12}z_{21}u^p_{11} - \pi_1\lambda_{12}r_{22}z_{21}u^p_{11} - \pi_2\lambda_{21}r_{12}z_{21}u^p_{12} - \pi_2\lambda_{22}r_{22}z_{21}u^p_{12}).
\end{aligned}
$$

By substituting the entries of (R_1, Z_1) and (R_2, Z_2) into the above equations, we obtain $r_{ij} = 0$ and $z_{ji} = 0$ for each $i, j \in (1, 2)$. Thus, this system has rest points at the partition equilibria (R_1, Z_1) and (R_2, Z_2).

Sequentially, we check dynamical stability of the partition equilibrium (Z_1, R_1). The characteristic equation of the Jacobian matrix evaluated at the rest point (Z_1, R_1) is given by

$(\lambda + \pi_1\lambda_{11}u^a_{11} + \pi_2\lambda_{21}u^a_{12})(\lambda - \pi_1\lambda_{11}u^a_{21} - \pi_2\lambda_{21}u^a_{22} + \pi_1\lambda_{11}u^a_{11} + \pi_2\lambda_{21}u^a_{12})(\lambda - \pi_1\lambda_{12}u^a_{11} - \pi_2\lambda_{22}u^a_{12} + \pi_1\lambda_{12}u^a_{21} + \pi_2\lambda_{22}u^a_{22})(\lambda + \pi_1\lambda_{12}u^a_{21} + \pi_2\lambda_{22}u^a_{22})(\lambda + \pi_1\lambda_{11}u^p_{11} + \pi_2\lambda_{21}u^p_{12})(\lambda - \pi_1\lambda_{11}u^p_{21} - \pi_2\lambda_{21}u^p_{22} + \pi_1\lambda_{11}u^p_{11} + \pi_2\lambda_{21}u^p_{12})(\lambda - \pi_1\lambda_{12}u^p_{11} - \pi_2\lambda_{22}u^p_{12} + \pi_1\lambda_{12}u^p_{21} + \pi_2\lambda_{22}u^p_{22})(\lambda + \pi_1\lambda_{12}u^p_{21} + \pi_2\lambda_{22}u^p_{22}) = 0$, where λ is the eigenvalue.

Thus, this system is structurally stable when

$$
\begin{aligned}
&\pi_1\lambda_{11}u^a_{21} - \pi_1\lambda_{11}u^a_{11} + \pi_2\lambda_{21}u^a_{22} - \pi_2\lambda_{21}u^a_{12} < 0, \\
&\pi_1\lambda_{12}u^a_{11} - \pi_1\lambda_{12}u^a_{21} + \pi_2\lambda_{22}u^a_{12} - \pi_2\lambda_{22}u^a_{22} < 0, \\
&\pi_1\lambda_{11}u^p_{21} - \pi_1\lambda_{11}u^p_{11} + \pi_2\lambda_{21}u^p_{22} - \pi_2\lambda_{21}u^p_{12} < 0, \\
&\pi_1\lambda_{12}u^p_{11} - \pi_1\lambda_{12}u^p_{21} + \pi_2\lambda_{22}u^p_{12} - \pi_2\lambda_{22}u^p_{22} < 0.
\end{aligned}
$$

Sequentially, we check the dynamical stability of the partition equilibrium (Z_2, R_2). The characteristic equation of the Jacobian matrix evaluated at the rest point (Z_2, R_2) has eight eigenvalues. One of them is zero. Thus, this system is structually unstable.

Proof of Theorem 5.2

A determinate action equilibrium is represented by the form:

$$R_3 = \begin{pmatrix} 1-\alpha & \alpha \\ 0 & 1 \end{pmatrix}, Z_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By substituting the entries of (R_3, Z_3) into our dynamical system $S' = \Phi'(S)$, we obtain $r_{ij} = 0$ and $z_{ji} = 0$ for each $i, j \in (1, 2)$;

$$\begin{cases} \dot{r}_{11} &= (1-\alpha)(\alpha\pi_1\lambda_{11}u^a_{11} + \alpha\pi_2\lambda_{21}u^a_{12} - \alpha\pi_1\lambda_{11}u^a_{21} - \alpha\pi_2\lambda_{21}u^a_{22}) = 0, \\ \dot{r}_{12} &= \alpha((1-\alpha)\pi_1\lambda_{11}u^a_{21} - (1-\alpha)\pi_1\lambda_{11}u^a_{11} + (1-\alpha)\pi_2\lambda_{21}u^a_{22} - (1-\alpha)\pi_2\lambda_{21}u^a_{12}) = 0, \\ \dot{r}_{21} &= 0, \\ \dot{r}_{22} &= 0, \\ \dot{z}_{11} &= 0, \\ \dot{z}_{12} &= 0, \\ \dot{z}_{21} &= 0, \\ \dot{z}_{22} &= 0. \end{cases}$$

Thus, this system has the rest point when $\pi_1\lambda_{11}u^a_{11} + \pi_2\lambda_{21}u^a_{12} - \pi_1\lambda_{11}u^a_{21} - \pi_2\lambda_{21}u^a_{22} = 0$.

Next, we check the dynamical stability of the determinate action equilibrium (Z_3, R_3). The characteristic equation of the Jacobian matrix evaluated at the rest point (Z_3, R_3) is given by $(\lambda - (2\alpha-1)\pi_1\lambda_{11}u^a_{11} - (2\alpha-1)\pi_2\lambda_{21}u^a_{12} + \alpha\pi_1\lambda_{11}u^a_{21} + \alpha\pi_2\lambda_{21}u^a_{22})(\lambda - (1-2\alpha)\pi_1\lambda_{11}u^a_{21} - (1-2\alpha)\pi_2\lambda_{21}u^a_{22} + (1-\alpha)\pi_1\lambda_{11}u^a_{11} + (1-\alpha)\pi_2\lambda_{21}u^a_{12})(\lambda - \pi_1\lambda_{12}u^a_{11} - \pi_2\lambda_{22}u^a_{12} + \pi_1\lambda_{12}u^a_{21} + \pi_2\lambda_{22}u^a_{22})(\lambda + \pi_1\lambda_{12}u^a_{21} + \pi_2\lambda_{22}u^a_{22})(\lambda + (1-\alpha)\pi_1\lambda_{11}u^p_{11} + (1-\alpha)\pi_2\lambda_{21}u^p_{12})(\lambda - (1-\alpha)\pi_1\lambda_{11}u^p_{21} + (1-

$\alpha)\pi_1\lambda_{11}u^p_{11} - (1-\alpha)\pi_2\lambda_{21}u^p_{22} + (1-\alpha)\pi_2\lambda_{21}u^p_{12})(\lambda - \alpha\pi_1\lambda_{11}u^p_{11} + \alpha\pi_1\lambda_{11}u^p_{21} - \pi_1\lambda_{12}u^p_{11} + \pi_1\lambda_{12}u^p_{21} - \alpha\pi_2\lambda_{21}u^p_{12} + \alpha\pi_2\lambda_{21}u^p_{22} - \pi_2\lambda_{22}u^p_{12} + \pi_2\lambda_{22}u^p_{22})(\lambda + \pi_1\lambda_{11}u^p_{21} + \pi_1\lambda_{12}u^p_{21} + \pi_2\lambda_{21}u^p_{22} + \pi_2\lambda_{22}u^p_{22}) = 0.$

Thus, this system is structurally stable when

$$
\begin{aligned}
&(2\alpha-1)\pi_1\lambda_{11}u^a_{11} + (2\alpha-1)\pi_2\lambda_{21}u^a_{12} - \alpha\pi_1\lambda_{11}u^a_{21} - \alpha\pi_2\lambda_{21}u^a_{22}) < 0,\\
&(1-2\alpha)\pi_1\lambda_{11}u^a_{21} + (1-2\alpha)\pi_2\lambda_{21}u^a_{22} - (1-\alpha)\pi_1\lambda_{11}u^a_{11} - (1-\alpha)\pi_2\lambda_{21}u^a_{12}) < 0,\\
&\pi_1\lambda_{12}u^a_{11} - \pi_1\lambda_{12}u^a_{21} + \pi_2\lambda_{22}u^a_{12} - \pi_2\lambda_{22}u^a_{22} < 0,\\
&\pi_1\lambda_{11}(u^p_{21} - u^p_{11}) + \pi_2\lambda_{21}(u^p_{22} - u^p_{12}) < 0,\\
&\alpha\pi_1\lambda_{11}(u^p_{11} - u^p_{21}) + \pi_1\lambda_{12}(u^p_{11} - u^p_{21}) + \alpha\pi_2\lambda_{21}(u^p_{12} - u^p_{22}) - \pi_2\lambda_{22}(u^p_{12} - u^p_{22}) < 0.
\end{aligned}
$$

Proof of Theorem 5.3

A random action equilibrium is represented by the form:

$$
R_4 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, Z_4 = \begin{pmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{pmatrix}.
$$

By substituting the entries of (R_4, Z_4) into our dynamical system $S' = \Phi'(S)$, we obtain $r_{ij} = 0$ and $z_{ji} = 0$ for each $i, j \in (1, 2)$:

$$\begin{cases} \dot{r}_{11} &= \frac{1}{2}((\frac{1}{2}-\frac{1}{2}\alpha)\pi_1\lambda_{11}u^a_{11}+(\frac{1}{2}-\frac{1}{2}\alpha)\pi_2\lambda_{21}u^a_{12} \\ & \quad -(\frac{1}{2}-\frac{1}{2}\alpha)\pi_1\lambda_{11}u^a_{21}-(\frac{1}{2}-\frac{1}{2}\alpha)\pi_2\lambda_{21}u^a_{22})=0, \\ \dot{r}_{12} &= \frac{1}{2}((\frac{1}{2}-\frac{1}{2}\alpha)\pi_1\lambda_{11}u^a_{11}+(\frac{1}{2}-\frac{1}{2}\alpha)\pi_2\lambda_{21}u^a_{12} \\ & \quad -(\frac{1}{2}-\frac{1}{2}\alpha)\pi_1\lambda_{11}u^a_{21}-(\frac{1}{2}-\frac{1}{2}\alpha)\pi_2\lambda_{21}u^a_{22})=0, \\ \dot{r}_{21} &= 0, \\ \dot{r}_{22} &= 0, \\ \dot{z}_{11} &= 0, \\ \dot{z}_{12} &= 0, \\ \dot{z}_{21} &= \alpha(-\frac{1}{2}\pi_1\lambda_{11}u^p_{21}-\pi_1\lambda_{12}u^p_{21}-\frac{1}{2}\pi_2\lambda_{21}u^p_{22}-\pi_2\lambda_{22}u^p_{22} \\ & \quad +\frac{1}{2}\pi_1\lambda_{11}u^p_{11}+\pi_1\lambda_{12}u^p_{11}+\frac{1}{2}\pi_2\lambda_{21}u^p_{12}+\pi_2\lambda_{22}u^p_{12})=0, \\ \dot{z}_{22} &= (1-\alpha)(\frac{1}{2}\pi_1\lambda_{11}u^p_{21}+\pi_1\lambda_{12}u^p_{21}+\frac{1}{2}\pi_2\lambda_{21}u^p_{22}+\pi_2\lambda_{22}u^p_{22} \\ & \quad -\frac{1}{2}\pi_1\lambda_{11}u^p_{11}-\pi_1\lambda_{12}u^p_{11}-\frac{1}{2}\pi_2\lambda_{21}u^p_{12}-\pi_2\lambda_{22}u^p_{12})=0. \end{cases}$$

Thus, this system has the rest point at the random action equilibrium (R_4, Z_4) when $\pi_1\lambda_{11}u^a_{11}-\pi_1\lambda_{11}u^a_{21}+\pi_2\lambda_{21}u^a_{12}-\pi_2\lambda_{21}u^a_{22}=0$, $\frac{1}{2}\pi_1\lambda_{11}u^p_{21}-\frac{1}{2}\pi_1\lambda_{11}u^p_{11}+\pi_1\lambda_{12}u^p_{21}-\pi_1\lambda_{12}u^p_{11}+\frac{1}{2}\pi_2\lambda_{21}u^p_{22}-\frac{1}{2}\pi_2\lambda_{21}u^p_{12}+\pi_2\lambda_{22}u^p_{22}-\pi_2\lambda_{22}u^p_{12}=0$.

Next, we check the stability of the random action equilibrium (R_4, Z_4). The characteristic equation of the Jacobian matrix evaluated at the rest point (R_4, Z_4) has eight eigenvalues. One of them is zero. Thus, this system is structurally unstable.

Proof of Theorem 5.4

We consider the case in which there is no information structure. Our dynamical system of the replicator dynamics consists of 8 differential equations:

$$\begin{cases} \dot{r}_{11} &= r_{11}(\pi_1 z_{11}u^a_{11} + \pi_1 z_{12}u^a_{21} - \pi_1 r_{11}z_{11}u^a_{11} - \pi_1 r_{11}z_{12}u^a_{21} - \pi_1 r_{12}z_{21}u^a_{11} - \pi_1 r_{12}z_{22}u^a_{21}), \\ \dot{r}_{12} &= r_{12}(\pi_1 z_{21}u^a_{11} + \pi_1 z_{22}u^a_{21} - \pi_1 r_{12}z_{21}u^a_{11} - \pi_1 r_{12}z_{22}u^a_{21} - \pi_1 r_{11}z_{11}u^a_{11} - \pi_1 r_{11}z_{12}u^a_{21}), \\ \dot{r}_{21} &= r_{21}(\pi_2 z_{11}u^a_{12} + \pi_2 z_{12}u^a_{22} - \pi_2 r_{21}z_{11}u^a_{12} - \pi_2 r_{21}z_{12}u^a_{22} - \pi_2 r_{22}z_{21}u^a_{12} - \pi_2 r_{22}z_{22}u^a_{22}), \\ \dot{r}_{22} &= r_{22}(\pi_2 z_{21}u^a_{12} + \pi_2 z_{22}u^a_{22} - \pi_2 r_{22}z_{21}u^a_{12} - \pi_2 r_{22}z_{22}u^a_{22} - \pi_2 r_{21}z_{11}u^a_{12} - \pi_2 r_{21}z_{12}u^a_{22}), \\ \dot{z}_{11} &= z_{11}(\pi_1 r_{11}u^p_{11} + \pi_2 r_{21}u^p_{12} - \pi_1 r_{11}z_{11}u^p_{11} - \pi_2 r_{21}z_{11}u^p_{12} - \pi_1 r_{11}z_{12}u^p_{21} - \pi_2 r_{21}z_{12}u^p_{22}), \\ \dot{z}_{12} &= z_{12}(\pi_1 r_{11}u^p_{21} + \pi_2 r_{21}u^p_{22} - \pi_1 r_{11}z_{12}u^p_{21} - \pi_2 r_{21}z_{12}u^p_{22} - \pi_1 r_{11}z_{11}u^p_{11} - \pi_2 r_{21}z_{11}u^p_{12}), \\ \dot{z}_{21} &= z_{21}(\pi_1 r_{12}u^p_{11} + \pi_2 r_{22}u^p_{12} - \pi_1 r_{12}z_{22}u^p_{21} - \pi_2 r_{22}z_{22}u^p_{22} - \pi_1 r_{12}z_{21}u^p_{11} - \pi_2 r_{22}z_{21}u^p_{12}), \\ \dot{z}_{22} &= z_{22}(\pi_1 r_{12}u^p_{21} + \pi_2 r_{22}u^p_{22} - \pi_1 r_{12}z_{22}u^p_{21} - \pi_2 r_{22}z_{22}u^p_{22} - \pi_1 r_{12}z_{21}u^p_{11} - \pi_2 r_{22}z_{21}u^p_{12}). \end{cases}$$

By substituting the entries of (R_1, Z_1) into the above equations, we obtain $r_{ij} = 0$ and $z_{ji} = 0$ for each $i, j \in (1, 2)$. Thus, this system has rest points at the partition equilibrium (R_1, Z_1).

Sequentially, we study the dynamical stability of these rest points. The characteristic equation of the Jacobian matrix evaluated at the rest point (Z_1, R_1) is given by $(\lambda+\pi_1 u^a_{11})(\lambda-\pi_1 u^a_{21}+\pi_1 u^a_{11})(\lambda-\pi_2 u^a_{12}+\pi_2 u^a_{22})(\lambda+\pi_2 u^a_{22})(\lambda+\pi_1 u^p_{11})(\lambda-\pi_1 u^p_{22} + \pi_1 u^p_{11})(\lambda - \pi_2 u^p_{12} + \pi_2 u^p_{22})(\lambda + \pi_2 u^p_{22}) = 0$, where λ is the eigenvalue.

Thus, this system is structurally stable when $u^a_{21} - u^a_{11} < 0, u^a_{12} - u^a_{22} < 0, u^p_{22} - u^p_{11} < 0$, and $u^p_{12} - u^p_{22} < 0$.

Proof of Theorem 5.5

As with Theorem 5.4, by substituting the entries of (R_4, Z_4) into our dynamical system of Proof of Theorem 5.4, we obtain $r_{ij} = 0$ and $z_{ji} = 0$ for each $i, j \in (1, 2)$:

$$\begin{cases} \dot{r}_{11} &= \pi_1 u^a_{11} - \pi_1(1-\alpha)u^a_{11} - \pi_1\alpha u^a_{21} = 0, \\ \dot{r}_{12} &= \pi_1 u^a_{21} - \pi_1\alpha u^a_{21} - \pi_1(1-\alpha)u^a_{11} = 0, \\ \dot{r}_{22} &= 0, \\ \dot{z}_{11} &= 0, \\ \dot{z}_{12} &= 0, \\ \dot{z}_{21} &= 0, \\ \dot{z}_{22} &= 0. \end{cases}$$

Thus, this system has the rest point at the random action equilibrium (R_4, Z_4) when $u^a_{11} = u^a_{21}$.

Next, we check the stability of the random action equilibrium (R_4, Z_4). The characteristic equation of the Jacobian matrix evaluated at the rest point (R_4, Z_4) is given by

$(\lambda + \alpha u^a_{21} - (1-2\alpha)u^a_{11})(\lambda - (1-2\alpha)u^a_{21} + (1-\alpha)u^a_{11})(\lambda - \pi_1 u^a_{12} + \pi_2 u^a_{22})(\lambda + \pi_2 u^a_{22})(\lambda + (1-\alpha)\pi_1 u^p_{11})(\lambda + (1-\alpha)\pi_1 u^p_{21} + (1-\alpha)\pi_1 u^p_{11})(\lambda - \pi_1\alpha u^p_{11} - \pi_2 u^p_{12} + \pi_1\alpha u^p_{21} + \pi_2 u^p_{22})(\lambda + \pi_1\alpha u^p_{21} + \pi_2 u^p_{22}) = 0$, where λ is the eigenvalue.

Thus, this system can be structurally stable when $-\alpha u^a_{21} + (1-2\alpha)u^a_{11} < 0, (1-2\alpha)u^a_{21} - (1-\alpha)u^a_{11} < 0, \pi_2 u^a_{12} - \pi_2 u^a_{22} < 0, u^p_{21} - u^p_{11} < 0, \pi_1\alpha u^p_{11} + \pi_2 u^p_{12} - \pi_1\alpha u^p_{21} - \pi_2 u^p_{22} < 0$.

Proof of Theorem 6.1

Assuming that there is a rest point close to a partition equilibrium (R_1, Z_1), we write down the rest point as follows:

$$R_1 = \begin{pmatrix} 1-\varepsilon_1 & \varepsilon_1 \\ \varepsilon_2 & 1-\varepsilon_2 \end{pmatrix}, Z_1 = \begin{pmatrix} 1-\delta_1 & \delta_1 \\ \delta_2 & 1-\delta_2 \end{pmatrix}.$$

Our dynamical system $\dot{S} = \Phi(S)$ of the selection–mutation dynamics consists of 8 differential equations. [3] By substituting the entries $(\tilde{z}_{ij}, \tilde{r}_{ji})$ of (R_1^*, Z_1^*) into 8 differential equations, we obtain the following system:

$$
\begin{aligned}
\dot{r}_{11} &= (1-\varepsilon_1)(\pi_1\lambda_{11}(1-\delta_1)u_{11}^a + \pi_1\lambda_{11}\delta_1 u_{21}^a + \pi_2\lambda_{21}(1-\delta_1)u_{12}^a + \pi_2\lambda_{21}\delta_1 u_{22}^a \\
&\quad -\pi_1\lambda_{11}(1-\varepsilon_1)(1-\delta_1)u_{11}^a - \pi_1\lambda_{11}(1-\varepsilon_1)\delta_1 u_{21}^a - \pi_2\lambda_{21}(1-\varepsilon_1)(1-\delta_1)u_{12}^a - \pi_2\lambda_{21}(1-\varepsilon_1)\delta_1 u_{22}^a \\
&\quad -\pi_1\lambda_{11}\varepsilon_1\delta_2 u_{11}^a - \pi_1\lambda_{11}\varepsilon_1(1-\delta_2)u_{21}^a - \pi_2\lambda_{21}\varepsilon_1\delta_2 u_{12}^a - \pi_2\lambda_{21}\varepsilon_1(1-\delta_2)u_{22}^a) \\
&\quad +\varepsilon(1-2(1-\varepsilon_1)), \\
\dot{r}_{12} &= \varepsilon_1(\pi_1\lambda_{11}\delta_2 u_{11}^a + \pi_1\lambda_{11}(1-\delta_2)u_{21}^a + \pi_2\lambda_{21}\delta_2 u_{12}^a + \pi_2\lambda_{21}(1-\delta_2)u_{22}^a \\
&\quad -\pi_1\lambda_{11}\varepsilon_1\delta_2 u_{11}^a - \pi_1\lambda_{11}\varepsilon_1(1-\delta_2)u_{21}^a - \pi_2\lambda_{21}\varepsilon_1\delta_2 u_{12}^a - \pi_2\lambda_{21}\varepsilon_1(1-\delta_2)u_{22}^a \\
&\quad -\pi_1\lambda_{11}(1-\varepsilon_1)(1-\delta_1)u_{11}^a - \pi_1\lambda_{11}(1-\varepsilon_1)\delta_1 u_{21}^a - \pi_2\lambda_{21}(1-\varepsilon_1)(1-\delta_1)u_{12}^a - \pi_2\lambda_{21}(1-\varepsilon_1)\delta_1 u_{22}^a) \\
&\quad +\varepsilon(1-2\varepsilon_1), \\
\dot{r}_{21} &= \varepsilon_2(\pi_1\lambda_{12}(1-\delta_1)u_{11}^a + \pi_1\lambda_{12}\delta_1 u_{21}^a + \pi_2\lambda_{22}(1-\delta_1)u_{12}^a + \pi_2\lambda_{22}\delta_1 u_{22}^a \\
&\quad -\pi_1\lambda_{12}\varepsilon_2(1-\delta_1)u_{11}^a - \pi_1\lambda_{12}\varepsilon_2\delta_1 u_{21}^a - \pi_2\lambda_{22}\varepsilon_2(1-\delta_1)u_{12}^a - \pi_2\lambda_{22}\varepsilon_2\delta_1 u_{22}^a \\
&\quad -\pi_1\lambda_{12}(1-\varepsilon_2)\delta_2 u_{11}^a - \pi_1\lambda_{12}(1-\varepsilon_2)(1-\delta_2)u_{21}^a - \pi_2\lambda_{22}(1-\varepsilon_2)\delta_2 u_{12}^a - \pi_2\lambda_{22}(1-\varepsilon_2)(1-\delta_2)u_{22}^a) \\
&\quad +\varepsilon(1-2\varepsilon_2), \\
\dot{r}_{22} &= (1-\varepsilon_2)(\pi_1\lambda_{12}\delta_2 u_{11}^a + \pi_1\lambda_{12}(1-\delta_2)u_{21}^a + \pi_2\lambda_{22}\delta_2 u_{12}^a + \pi_2\lambda_{22}(1-\delta_2)u_{22}^a \\
&\quad -\pi_1\lambda_{12}(1-\varepsilon_2)\delta_2 u_{11}^a - \pi_1\lambda_{12}(1-\varepsilon_2)(1-\delta_2)u_{21}^a - \pi_2\lambda_{22}(1-\varepsilon_2)\delta_2 u_{12}^a - \pi_2\lambda_{22}(1-\varepsilon_2)(1-\delta_2)u_{22}^a \\
&\quad -\pi_1\lambda_{12}\varepsilon_2(1-\delta_1)u_{11}^a - \pi_1\lambda_{12}\varepsilon_2\delta_1 u_{21}^a - \pi_2\lambda_{22}\varepsilon_2(1-\delta_1)u_{12}^a - \pi_2\lambda_{22}\varepsilon_2\delta_1 u_{22}^a) \\
&\quad +\varepsilon(1-2(1-\varepsilon_2)), \\
\dot{z}_{11} &= (1-\delta_1)(\pi_1\lambda_{11}(1-\varepsilon_1)u_{11}^p + \pi_1\lambda_{12}\varepsilon_2 u_{11}^p + \pi_2\lambda_{21}(1-\varepsilon_1)u_{12}^p + \pi_2\lambda_{22}\varepsilon_2 u_{12}^p \\
&\quad -\pi_1\lambda_{11}(1-\varepsilon_1)(1-\delta_1)u_{11}^p - \pi_1\lambda_{12}\varepsilon_2(1-\delta_1)u_{11}^p - \pi_2\lambda_{21}(1-\varepsilon_1)(1-\delta_1)u_{12}^p - \pi_2\lambda_{22}\varepsilon_2(1-\delta_1)u_{12}^p \\
&\quad -\pi_1\lambda_{11}(1-\varepsilon_1)\delta_1 u_{21}^p - \pi_1\lambda_{12}\varepsilon_2\delta_1 u_{21}^p - \pi_2\lambda_{21}(1-\varepsilon_1)\delta_1 u_{22}^p - \pi_2\lambda_{22}\varepsilon_2\delta_1 u_{22}^p) \\
&\quad +\delta(1-2(1-\delta_1)), \\
\dot{z}_{12} &= \delta_1(\pi_1\lambda_{11}(1-\varepsilon_1)u_{21}^p + \pi_1\lambda_{12}\varepsilon_2 u_{21}^p + \pi_2\lambda_{21}(1-\varepsilon_1)u_{22}^p + \pi_2\lambda_{22}\varepsilon_2 u_{22}^p \\
&\quad -\pi_1\lambda_{11}(1-\varepsilon_1)\delta_1 u_{21}^p - \pi_1\lambda_{12}\varepsilon_2\delta_1 u_{21}^p - \pi_2\lambda_{21}(1-\varepsilon_1)\delta_1 u_{22}^p - \pi_2\lambda_{22}\varepsilon_2\delta_1 u_{22}^p \\
&\quad -\pi_1\lambda_{11}(1-\varepsilon_1)(1-\delta_1)u_{11}^p - \pi_1\lambda_{12}\varepsilon_2(1-\delta_1)u_{11}^p - \pi_2\lambda_{21}(1-\varepsilon_1)(1-\delta_1)u_{12}^p - \pi_2\lambda_{22}\varepsilon_2(1-\delta_1)u_{12}^p) \\
&\quad +\delta(1-2\delta_1), \\
\dot{z}_{21} &= \delta_2(\pi_1\lambda_{11}\varepsilon_1 u_{11}^p + \pi_1\lambda_{12}(1-\varepsilon_2)u_{11}^p + \pi_2\lambda_{21}\varepsilon_1 u_{12}^p + \pi_2\lambda_{22}(1-\varepsilon_2)u_{12}^p \\
&\quad -\pi_1\lambda_{11}\varepsilon_1(1-\delta_2)u_{21}^p - \pi_1\lambda_{12}(1-\varepsilon_2)(1-\delta_2)u_{21}^p - \pi_2\lambda_{21}\varepsilon_1(1-\delta_2)u_{22}^p - \pi_2\lambda_{22}(1-\varepsilon_2)(1-\delta_2)u_{22}^p \\
&\quad -\pi_1\lambda_{11}\varepsilon_1\delta_2 u_{11}^p - \pi_1\lambda_{12}(1-\varepsilon_2)\delta_2 u_{11}^p - \pi_2\lambda_{21}\varepsilon_1\delta_2 u_{12}^p - \pi_2\lambda_{22}(1-\varepsilon_2)\delta_2 u_{12}^p) \\
&\quad +\delta(1-2\delta_2), \\
\dot{z}_{22} &= (1-\delta_2)(\pi_1\lambda_{11}\varepsilon_1 u_{21}^p + \pi_1\lambda_{12}(1-\varepsilon_2)u_{21}^p + \pi_2\lambda_{21}\varepsilon_1 u_{22}^p + \pi_2\lambda_{22}(1-\varepsilon_2)u_{22}^p \\
&\quad -\pi_1\lambda_{11}\varepsilon_1(1-\delta_2)u_{21}^p - \pi_1\lambda_{12}(1-\varepsilon_2)(1-\delta_2)u_{21}^p - \pi_2\lambda_{21}\varepsilon_1(1-\delta_2)u_{22}^p - \pi_2\lambda_{22}(1-\varepsilon_2)(1-\delta_2)u_{22}^p \\
&\quad -\pi_1\lambda_{11}\varepsilon_1\delta_2 u_{11}^p - \pi_1\lambda_{12}(1-\varepsilon_2)\delta_2 u_{11}^p - \pi_2\lambda_{21}\varepsilon_1\delta_2 u_{12}^p - \pi_2\lambda_{22}(1-\varepsilon_2)\delta_2 u_{12}^p) \\
&\quad +\delta(1-2(1-\delta_2)).
\end{aligned}
$$

[3]See p 10 in this paper.

We remove redundant equations $\dot{r}_{ij}$ and $\dot{z}_{ji}$ for $i = j = 1$ and $i = j = 2$.

Let Df denote the Jacobian matrix of $\dot{r}_{ij}$ and $\dot{z}_{ji}$ with respect to $\varepsilon_1, \varepsilon_2, \delta_1$ and δ_2, that is,

$$Df = \begin{pmatrix} \frac{\partial \dot{r}_{12}}{\partial \varepsilon_1} & \frac{\partial \dot{r}_{12}}{\partial \varepsilon_2} & \frac{\partial \dot{r}_{12}}{\partial \delta_1} & \frac{\partial \dot{r}_{12}}{\partial \delta_2} \\ \frac{\partial \dot{r}_{21}}{\partial \varepsilon_1} & \frac{\partial \dot{r}_{21}}{\partial \varepsilon_2} & \frac{\partial \dot{r}_{21}}{\partial \delta_1} & \frac{\partial \dot{r}_{21}}{\partial \delta_2} \\ \frac{\partial \dot{z}_{12}}{\partial \varepsilon_1} & \frac{\partial \dot{z}_{12}}{\partial \varepsilon_2} & \frac{\partial \dot{z}_{12}}{\partial \delta_1} & \frac{\partial \dot{z}_{12}}{\partial \delta_2} \\ \frac{\partial \dot{z}_{12}}{\partial \varepsilon_1} & \frac{\partial \dot{z}_{21}}{\partial \varepsilon_2} & \frac{\partial \dot{z}_{21}}{\partial \delta_1} & \frac{\partial \dot{z}_{21}}{\partial \delta_2} \end{pmatrix}.$$

Let $\det(Df(\mathbf{x}))$ denote the determinant of $Df(\mathbf{x})$ at point $\mathbf{x} = (\varepsilon_1, \varepsilon_2, \delta_1, \delta_2; \varepsilon, \delta)$. Since

$$\begin{aligned} &Df(\mathbf{0}) \\ &= \begin{pmatrix} \pi_1\lambda_{11}u^a_{21} + \pi_2\lambda_{21}u^a_{22} - \pi_1\lambda_{11}u^a_{11} - \pi_2\lambda_{21}u^a_{12} & 0 & 0 & 0 \\ 0 & \pi_1\lambda_{12}u^a_{11} + \pi_2\lambda_{22}u^a_{12} - \pi_1\lambda_{12}u^a_{21} - \pi_2\lambda_{22}u^a_{22} & 0 & 0 \\ 0 & 0 & \pi_1\lambda_{11}u^p_{21} + \pi_2\lambda_{21}u^p_{22} - \pi_1\lambda_{11}u^p_{11} - \pi_2\lambda_{21}u^p_{12} & 0 \\ 0 & 0 & 0 & \pi_1\lambda_{12}u^p_{11} + \pi_2\lambda_{22}u^p_{12} - \pi_1\lambda_{12}u^p_{21} - \pi_2\lambda_{22}u^p_{22} \end{pmatrix} \end{aligned}$$

at the point $(\varepsilon_1, \varepsilon_2, \delta_1, \delta_2; \varepsilon, \delta) = (0, 0, 0, 0; 0, 0)$, we have $Df(0) \neq 0$ when

$$\begin{aligned} &\pi_1\lambda_{11}u^a_{21} + \pi_2\lambda_{21}u^a_{22} - \pi_1\lambda_{11}u^a_{11} - \pi_2\lambda_{22}u^a_{12} \neq 0, \\ &\pi_1\lambda_{12}u^a_{11} + \pi_2\lambda_{22}u^a_{12} - \pi_1\lambda_{12}u^a_{21} - \pi_2\lambda_{22}u^a_{22} \neq 0, \\ &\pi_1\lambda_{11}u^p_{21} + \pi_2\lambda_{21}u^p_{22} - \pi_1\lambda_{11}u^p_{11} - \pi_2\lambda_{21}u^p_{12} \neq 0, \\ &\pi_1\lambda_{12}u^p_{11} + \pi_2\lambda_{22}u^p_{12} - \pi_1\lambda_{12}u^p_{21} - \pi_2\lambda_{22}u^p_{22} \neq 0. \end{aligned}$$

Proof of Corollary 1

Tayler's formula for the function $(\varepsilon_1(\varepsilon, \delta), \varepsilon_2(\varepsilon, \delta), \delta_1(\varepsilon, \delta), \delta_2(\varepsilon, \delta))$ about $(\varepsilon, \delta) = (0, 0)$ is given by

$$\begin{pmatrix} \varepsilon_1(\varepsilon, \delta) \\ \varepsilon_2(\varepsilon, \delta) \\ \delta_1(\varepsilon, \delta) \\ \delta_2(\varepsilon, \delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(0,0) \\ \varepsilon_2(0,0) \\ \delta_1(0,0) \\ \delta_2(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_1}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_2}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_2}{\partial \delta}(0,0) \\ \frac{\partial \delta_1}{\partial \varepsilon}(0,0) & \frac{\partial \delta_1}{\partial \delta}(0,0) \\ \frac{\partial \delta_2}{\partial \varepsilon}(0,0) & \frac{\partial \delta_2}{\partial \delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon, \delta) \\ o_2(\varepsilon, \delta) \\ o_3(\varepsilon, \delta) \\ o_4(\varepsilon, \delta) \end{pmatrix}.$$

Because $(\varepsilon_1(0,0),\varepsilon_2(0,0),\delta_1(0,0),\delta_2(0,0))$ is a solution of the system $f_I(\varepsilon_1(0,0),\varepsilon_2(0,0),\delta_1(0,0),\delta_2(0,0);0,0)=0, I=1,2,3,4$, we obtain $(\varepsilon_1(0,0),\varepsilon_2(0,0),\delta_1(0,0),\delta_2(0,0))=(0,0,0,0)$.

By the implicit function theorem and the fact that

$$Df(\mathbf{0})=$$

$$\begin{pmatrix} \pi_1\lambda_{11}u^a_{21}+\pi_2\lambda_{21}u^a_{22}-\pi_1\lambda_{11}u^a_{11}-\pi_2\lambda_{21}u^a_{12} & 0 & 0 & 0 \\ 0 & \pi_1\lambda_{12}u^a_{11}+\pi_2\lambda_{22}u^a_{12}-\pi_1\lambda_{12}u^a_{21}-\pi_2\lambda_{22}u^a_{22} & 0 & 0 \\ 0 & 0 & \pi_1\lambda_{11}u^p_{21}+\pi_2\lambda_{21}u^p_{22}-\pi_1\lambda_{11}u^p_{11}-\pi_2\lambda_{21}u^p_{12} & 0 \\ 0 & 0 & 0 & \pi_1\lambda_{12}u^p_{11}+\pi_2\lambda_{22}u^p_{12}-\pi_1\lambda_{12}u^p_{21}-\pi_2\lambda_{22}u^p_{22} \end{pmatrix}.$$

We obtain

$$\begin{pmatrix} \frac{\partial\varepsilon_1}{\partial\varepsilon}(0,0) & \frac{\partial\varepsilon_1}{\partial\delta}(0,0) \\ \frac{\partial\varepsilon_2}{\partial\varepsilon}(0,0) & \frac{\partial\varepsilon_2}{\partial\delta}(0,0) \\ \frac{\partial\delta_1}{\partial\varepsilon}(0,0) & \frac{\partial\delta_1}{\partial\delta}(0,0) \\ \frac{\partial\delta_2}{\partial\varepsilon}(0,0) & \frac{\partial\delta_2}{\partial\delta}(0,0) \end{pmatrix} = -(Df(\mathbf{0})^{-1}) \begin{pmatrix} \frac{\partial f_1}{\partial\varepsilon}(\mathbf{0}) & \frac{\partial f_1}{\partial\delta}(\mathbf{0}) \\ \frac{\partial f_2}{\partial\varepsilon}(\mathbf{0}) & \frac{\partial f_2}{\partial\delta}(\mathbf{0}) \\ \frac{\partial f_3}{\partial\varepsilon}(\mathbf{0}) & \frac{\partial f_3}{\partial\delta}(\mathbf{0}) \\ \frac{\partial f_4}{\partial\varepsilon}(\mathbf{0}) & \frac{\partial f_4}{\partial\delta}(\mathbf{0}) \end{pmatrix}$$

$$= -(Df(\mathbf{0})^{-1})^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\pi_1\lambda_{11}u^a_{11}+\pi_2\lambda_{21}u^a_{12}-\pi_1\lambda_{11}u^a_{21}-\pi_2\lambda_{21}u^a_{22}} & 0 \\ \frac{1}{\pi_1\lambda_{12}u^a_{21}+\pi_2\lambda_{22}u^a_{22}-\pi_1\lambda_{12}u^a_{11}-\pi_2\lambda_{22}u^a_{12}} & 0 \\ 0 & \frac{1}{\pi_1\lambda_{11}u^p_{11}+\pi_2\lambda_{21}u^p_{12}-\pi_1\lambda_{11}u^p_{21}-\pi_2\lambda_{21}u^p_{22}} \\ 0 & \frac{1}{\pi_1\lambda_{12}u^p_{21}+\pi_2\lambda_{22}u^p_{22}-\pi_1\lambda_{12}u^p_{11}-\pi_2\lambda_{22}u^p_{12}} \end{pmatrix},$$

where $\mathbf{0} = (0,0,0,0;0,0)$. Thus, Tayler's formula described above becomes

$$\begin{pmatrix} \varepsilon_1(\varepsilon,\delta) \\ \varepsilon_2(\varepsilon,\delta) \\ \delta_1(\varepsilon,\delta) \\ \delta_2(\varepsilon,\delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(0,0) \\ \varepsilon_2(0,0) \\ \delta_1(0,0) \\ \delta_2(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_1}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_2}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_2}{\partial \delta}(0,0) \\ \frac{\partial \delta_1}{\partial \varepsilon}(0,0) & \frac{\partial \delta_1}{\partial \delta}(0,0) \\ \frac{\partial \delta_2}{\partial \varepsilon}(0,0) & \frac{\partial \delta_2}{\partial \delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \\ o_4(\varepsilon,\delta) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\pi_1\lambda_{11}u^a_{11}+\pi_2\lambda_{21}u^a_{12}-\pi_1\lambda_{11}u^a_{21}-\pi_2\lambda_{21}u^a_{22}} & 0 \\ \frac{1}{\pi_1\lambda_{12}u^a_{21}+\pi_2\lambda_{22}u^a_{22}-\pi_1\lambda_{12}u^a_{11}-\pi_2\lambda_{22}u^a_{12}} & 0 \\ 0 & \frac{1}{\pi_1\lambda_{11}u^p_{11}+\pi_2\lambda_{21}u^p_{12}-\pi_1\lambda_{11}u^p_{21}-\pi_2\lambda_{21}u^p_{22}} \\ 0 & \frac{1}{\pi_1\lambda_{12}u^p_{21}+\pi_2\lambda_{22}u^p_{22}-\pi_1\lambda_{12}u^p_{11}-\pi_2\lambda_{22}u^p_{12}} \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix}$$

$$+ \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \\ o_4(\varepsilon,\delta) \end{pmatrix},$$

where $o_I(\varepsilon,\delta)$, $I = 1,2,3,4$, stands for the second- or higher-order terms of ε and δ. Thus, we obtain the first-order approximated values of $\varepsilon_1, \varepsilon_2$, δ_1 and δ_2, respectively, as follows:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{\pi_1\lambda_{11}u^a_{11}+\pi_2\lambda_{21}u^a_{12}-\pi_1\lambda_{11}u^a_{21}-\pi_2\lambda_{21}u^a_{22}}\varepsilon + o_1(\varepsilon,\delta), \\ \varepsilon_2 &= \frac{1}{\pi_1\lambda_{12}u^a_{21}+\pi_2\lambda_{22}u^a_{22}-\pi_1\lambda_{12}u^a_{11}-\pi_2\lambda_{22}u^a_{12}}\varepsilon + o_2(\varepsilon,\delta), \\ \delta_1 &= \frac{1}{\pi_1\lambda_{11}u^p_{11}+\pi_2\lambda_{21}u^p_{12}-\pi_1\lambda_{11}u^p_{21}-\pi_2\lambda_{21}u^p_{22}}\delta + o_3(\varepsilon,\delta), \\ \delta_2 &= \frac{1}{\pi_1\lambda_{12}u^p_{21}+\pi_2\lambda_{22}u^p_{22}-\pi_1\lambda_{12}u^p_{11}-\pi_2\lambda_{22}u^p_{12}}\delta + o_4(\varepsilon,\delta). \end{aligned}$$

We find the first-order approximated rest point. □

Proof of Theorem 6.2

Suppose that $\Lambda = I, \pi_1 = \pi_2 = \frac{1}{2}$, $U^A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, and $U^P = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \end{pmatrix}$.

In this case, the characteristic equation of the first-order approximated Jacobian matrix evaluated at the rest point close to the partition equilibrium is given by $(\lambda - \varepsilon - 2\delta + \frac{1}{2})(\lambda + \varepsilon - \frac{9}{2}\delta + \frac{1}{2})(\lambda + \frac{3}{4}\varepsilon - \frac{5}{4}\delta + \frac{1}{4})(\lambda - \frac{1}{2}\varepsilon + 4\delta + \frac{1}{4})(\lambda - \varepsilon - 2\delta + \frac{1}{2})(\lambda - \varepsilon + 1)(\lambda + \frac{1}{2}\varepsilon)(\lambda + \frac{1}{2}\varepsilon + \frac{1}{2}\delta + \frac{1}{2})^2 = 0$, where λ is the eigenvalue. Thus, this system is structurally stable.

Without perturbations, $\varepsilon = \delta = 0$. Thus, this characteristic equation has zero eigenvalue. It is structurally unstable.

References

[1] Green, J., and Stokey, N., A two-person game of informationtransmission. *Journal of Economic Theory*, 135:90-104, 2007.

[2] Hofbauer, J., The selection mutation equation. *Journal of Mathematical Biology*, 23(1):41-53, 1985.

[3] Hofbauer, J., Huttegger, S., Feasibility of communication in binary signaling. *Journal of Theoretical Biology*, 254:843-849, 2008.

[4] Hofbauer, J., Huttegger, S., Selection-mutation dynamics of signaling games. *Games*, 6(1):2-31, 2015.

[5] Hofbauer, J.; Sigmund, K. Evolutionary Games and Population Dynamics; Cambridge University Press: Cambridge, UK, 1998.

[6] Uchida, S., and Fukuzumi, M., Efficiency and stability in sender-receiver games under the selection-mutation dynamics. *TCER Working Paper Series*, E132, 2019.