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Quantization of Hamiltonian loop group spaces. (English) Zbl 1416.81068

Math. Ann. 374, No. 1-2, 681-722 (2019).

Let G be a a compact connected Lie group with its Lie algebra $\mathfrak{g}$, LG the loop group, and $(\mathcal{M}, \omega_{\mathcal{M}}, \Phi_{\mathcal{M}})$ a Hamiltonian LG-space with proper moment map $\Phi_{\mathcal{M}} : \mathcal{M} \to L\mathfrak{g}^*$.

This paper describes and studies an index-theoretic *quantization* of $\mathcal{M}$, in the spirit of the quantization of Hamiltonian G-space via the equivariant index of twisted Dirac operator, and the $[Q, R] = 0$ theorem [*V. Guillemin* and *S. Sternberg*, Invent. Math. 67, 515–538 (1982; Zbl 0503.58018); *P. Hochs* and *Y. Song*, Duke Math. J. 166, No. 6, 1125–1178 (2017; Zbl 1370.58010); *E. Meinrenken*, Adv. Math. 134, No. 2, 240–277 (1998; Zbl 0929.53045); *P.-E. Paradan*, J. Funct. Anal. 187, No. 2, 442–509 (2001; Zbl 1001.53062); *P.-E. Paradan* and *M. Vergne*, Acta Math. 218, No. 1, 137–199 (2017; Zbl 1385.53035); *Y. Tian* and *W. Zhang*, Invent. Math. 132, No. 2, 229–259 (1998; Zbl 0944.53047)]. In ["Spinor modules for Hamiltonian loop group spaces", Preprint, `arXiv:1706.07493`], the authors, together with Meinrenken, constructed a finite-dimensional *global transversal* $\mathcal{Y} \subset \mathcal{M}$ as well as a canonical spinor module $S \to \mathcal{Y}$.

This paper demonstrates that the corresponding spin-c Dirac operator $\mathcal{D}$ acting on sections of S twisted by L determines an element $[\mathcal{D}]$ in a suitable K-homology group. Taking the *index pairing* or *cap product* with a suitable K-cohomology class x, an element $\mathrm{x} \cap [\mathcal{D}]$ of the formal completion of the representation ring $R^{-\infty}(T)$ of a maximal torus $T \subset G$ is obtained. The corresponding multiplicity function is anti-symmetric under the action of the affine Weyl group, meaning that it is the numerator of the Weyl-Kac character formula of a graded *positive energy representation*, which is taken as $Q(\mathcal{M}, L)$.

In ["Geometric K-homology and the Freed-Hopkins-Teleman theorem", Preprint, `arXiv:1804.05213`], the first author established that $Q(\mathcal{M}, L)$ coincides with the image under the Freed-Hopkins-Teleman isomorphism [*D. S. Freed* et al., Ann. Math. (2) 174, No. 2, 947–1007 (2011; Zbl 1239.19002); *D. S. Freed* et al., J. Topol. 4, No. 4, 737–798 (2011; Zbl 1241.19002); *D. S. Freed* et al., J. Am. Math. Soc. 26, No. 3, 595–644 (2013; Zbl 1273.22015)] of the quantization of $M = \mathcal{M}/\Omega G$ in terms of twisted K-homology. This relationship is discussed briefly in §4.8. One fascinating feature of $Q(\mathcal{M}, L)$ in this paper is its amenability to the Witten deformation [Zbl 1001.53062; *P.-E. Paradan* and *M. Vergne*, Witten non abelian localization for equivariant K-theory, and the $[Q, R] = 0$ theorem. Providence, RI: American Mathematical Society (AMS) (2019; Zbl 07127346); Zbl 0944.53047]. In ["Norm-square localization and the quantization of Hamiltonian loop group spaces", Preprint, `arXiv:1810.02347`], the authors studied this deformation in detail, showing a formula in the spirit of [Zbl 1001.53062]:

$$\mathrm{x} \cap [\mathcal{D}] = \sum_{\beta \in W \cdot \mathcal{B}} \mathrm{index}(\sigma_{\beta,\theta} \otimes \mathrm{Sym}(\upsilon_\beta))$$

where x is a suitable K-cohomology class, $\mathcal{B} \subset \mathfrak{t}_+$ indexes components of the critical set of the norm-square of the moment map, $\sigma_{\beta,\theta}$ is a transversally elliptic symbol on the fixed-point set $\mathcal{Y}^\beta$, and υ_β is the normal bundle to $\mathcal{Y}^\beta$ in $\mathcal{Y}$. endowed with a β-*polarized* complex structure.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

81Q35 Quantum mechanics on special spaces: manifolds, fractals, graphs, lattices

81Q05 Closed and approximate solutions to the Schrödinger, Dirac, Klein-Gordon and other equations of quantum mechanics

81R20 Covariant wave equations in quantum theory, relativistic quantum mechanics

22E67 Loop groups and related constructions, group-theoretic treatment

55P47 Infinite loop spaces

Full Text: DOI arXiv

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