Rivera, Manuel; Zeinalian, Mahmoud

The colimit of an ∞-local system as a twisted tensor product. (English) Zbl 07173316

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The principal objective in this paper is to give an explicit model for the homotopy coherent colimit of an ∞-local system of chain complexes over a topological space in terms of Brown's tensor product construction [*E. H. Brown jun.*, Ann. Math. (2) 69, 223–246 (1959; Zbl 0199.58201)]. The framework of quasi-categories is used to describe three equivalent ∞-categories of ∞-local systems and one of these models is exploited to calculate the desired colimit. The main application of the paper is, the authors say, to L-theory through [*A. Ranicki* and *M. Weiss*, Math. Gottingensis, Schriftenr. Sonderforschungsbereichs Geom. Anal. 28, 48 p. (1987; Zbl 0616.55016); Math. Z. 204, No. 2, 157–185 (1990; Zbl 0669.55010)] where the category of fractured complexes with Poincaré duality is depicted as classical local systems. It is the authors' eventual goal to replace the fundamental group ring in Ranicki and Weiss's discussion with an algebraic model for the chains on the based loop space in hope of combining the algebraic theory of homotopy types à la Sullivan, Quillen and Mandell with the algebraic L-theory à la Ranicki and Weiss and to obtain a purely algebraic characterization of manifolds. Another possible application of the paper is to describe a model for the colimit of local systems of dg categories coming up in the discussion of Mirror Symmetry through generalizations of the main results of the paper.

Now we recall the classical legend, of which this paper is a generalization. Let $\boldsymbol{k}$ be a field, $Cat_{\boldsymbol{k}}$ the category of $\boldsymbol{k}$-linear categories, Cat the category of ordinary categories,

$$U : Cat_{\boldsymbol{k}} \to Cat$$

the forgetful functor and

$$F : Cat \to Cat_{\boldsymbol{k}}$$

its left adjoint. A representation of a group G is no other than a functor

$$\beta : G \to U(\boldsymbol{k}\text{-mod})$$

where G is to be thought of as a category with a single object denoted by b, and $\boldsymbol{k}$-mod $\in Cat_{\boldsymbol{k}}$ is the $\boldsymbol{k}$-linear category of $\boldsymbol{k}$-vector spaces. By adjunction, we get a functor

$$\widetilde{\beta} : F(G) \to \boldsymbol{k}\text{-mod}$$

$F(G) = \boldsymbol{k}[G]$ is the group algebra of G, thought of a $\boldsymbol{k}$-linear category with the single object b, and $\widetilde{\beta}(b) = M$ is a left $\boldsymbol{k}[G]$-module. The colimit of the functor β is the $\boldsymbol{k}$-module of coinvariants

$$\boldsymbol{k} \otimes_{\boldsymbol{k}[G]} M$$

where the right $\boldsymbol{k}[G]$-module structure on is given by the augmentation

$$\boldsymbol{k}[G] \to \boldsymbol{k}$$

The homotopy colimit of the functor

$$i \circ \beta : G \to U(\boldsymbol{k}\text{-mod}) \to Ch_{\boldsymbol{k}}$$

with $Ch_{\boldsymbol{k}}$ being the category of $\boldsymbol{k}$-chain complexes endowed with the standard model structure and $i : U(\boldsymbol{k}\text{-mod}) \to Ch_{\boldsymbol{k}}$ being the inclusion functor is the derived coinvariants

$$\boldsymbol{k} \otimes^{\mathbb{L}}_{\boldsymbol{k}[G]} M$$

and a model for it is to be obtained by resolving $\boldsymbol{k}$ through the bar resolution over $\boldsymbol{k}[G]$. In case that G is the fundamental group $\pi_1(X, b)$ of a pointed path-connected space (X, b), representations of G are classical local systems over X, where the colimit of a representation of G is no other than the homology with local coefficients and the homotopy limit is to be interpreted as a chain complex, unique up to quasi-isomorphism, calculating such homology groups.

The authors generalize the above constructions and results to ∞-representations of ∞-groupoids a.k.a. ∞-local systems by replacing Cat by the category Set_Δ of simplicial sets and the $\boldsymbol{k}$-linear category $\boldsymbol{k}$-mod by the differential graded dg category $Ch_{\boldsymbol{k}}$ of $\boldsymbol{k}$-chain complexes, where the analogue of U now becomes the dg nerve functor

$$N_{\mathrm{dg}} : dgCat_{\boldsymbol{k}} \to Set_\Delta$$

with $dgCat_{\boldsymbol{k}}$ being the ordinary category of dg categories, and the analogue of F is a functor

$$\Lambda : Set_\Delta \to dgCat_{\boldsymbol{k}}$$

explained in §4. AS was depicted in [Zbl 1423.18026], for a connected Kan complex K, $\Lambda(K)$ is closely related to the dg algebra of singular chains on the based Moore loop space of $|K|$.

The authors replace the fundamental group $\pi_1(X, b)$ by the Kan complex $\mathrm{Sing}(X)$ of singular simplices in X, which is equivalent to the Kan complex $\mathrm{Sing}(X, b)$ with a single 0-simplex if X is path-connected and $b \in X$. Representations

$$\beta : G \to U(\boldsymbol{k}\text{-mod})$$

are replaced by maps of simplicial sets

$$\beta : \mathrm{Sing}(X, b) \to N_{\mathrm{dg}} Ch_{\boldsymbol{k}}$$

which is equivalent, thanks to adjunction, to a dg functor

$$\widetilde{\beta} : \Lambda(\mathrm{Sing}(X, b)) \to Ch_{\boldsymbol{k}}$$

being interpreted as a chain complex

$$\widetilde{\beta}(b) = M$$

endowed with an action over the dg algebra

$$\Lambda(\mathrm{Sing}(X, b))(b, b)$$

This paper gives a model for the colimit of β by using the framework of quasi-categories of [*J. Lurie*, Higher topos theory. Princeton, NJ: Princeton University Press (2009; Zbl 1175.18001)], where the term *colimit* is to be understood in the homotopy coherent sense of §1.2.13 in [Zbl 1175.18001]. It was established in [*M. Rivera* and *M. Zeinalian*, Algebr. Geom. Topol. 18, No. 7, 3789–3820 (2018; Zbl 1423.18026)] that $\Lambda(\mathrm{Sing}(X, b))(b, b)$ is isomorphic as a dg algebra to ΩC, the cobar construction of the dg coalgebra C of normalized chains on $\mathrm{Sing}(X, b)$ with Alexander-Whitney coproduct, opening up the possibility of using certain algebraic gadgets to study ∞-local systems.

A brief synopsis of the paper consisting of 8 sections is now in order. Some preliminary notions are gone over in §2 so as to keep the paper as self-contained and accessible to a wider audience as possible. §3 reviews the rigidification functor [Zbl 1175.18001] and its cubical version [Zbl 1423.18026], which are useful for understanding the dg nerve functor N_{dg} and its adjoint Λ from a radically different viewpoint. §4 introduces N_{dg} and Λ, and investigates some of their properties.

Using quasi-categories as models for ∞-categories, it is shown that the ∞-categories of ∞-local systems Loc_X^∞ is equivalent to the ∞-derived category of dg ΩC-modules, where Loc_X^∞ is defined as the quasi-category of functors

$$\mathrm{Fun}(\mathrm{Sing}(X, b), N_{\mathrm{dg}} Ch_{\boldsymbol{k}})$$

Two equivalent quasi-categorical models for the ∞-derived category of dg ΩC-modules are given by taking the dg nerve of two dg categories $\mathrm{Mod}_{\Omega C}^\infty$ and $\mathrm{Mod}_{\Omega C}^\tau$ introduced in §5.2 and §5.3 respectively. It is shown in §6 that

$$N_{\mathrm{dg}} \mathrm{Mod}_{\Omega C}^\infty \simeq N_{\mathrm{dg}} \mathrm{Mod}_{\Omega C}^\tau \simeq Loc_X^\infty$$

The existence of a weak equivalence

$$Loc_X^\infty \simeq N_{\mathrm{dg}} \mathrm{Mod}_{\Omega C}^\infty$$

belongs to the folklore, but the authors give a direct proof based on the combinatorics of simplicial sets.

Another version of this equivalence also appeared in [*V. S. Braunack-Meyer*, Rational parametrised stable homotopy theory. Diss. Zürich, `https://user.math.uzh.ch/cattaneo/Braunack-Mayer.pdf`]. Koszul duality [*L. Positselski*, Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. Providence, RI: American Mathematical Society (AMS) (2011; Zbl 1275.18002)] implies that, for any conilpotent dg coalgebra C, the ∞-derived category of dg C-comodules is equivalent to that of dg ΩC-modules and, therefore, equivalent to Loc_X^∞ [*J. Chuang*, *J. Holstein* and *A. Lazarev*, "Maurer-Cartan moduli and theorems of Riemann-Hilbert type", Preprint, `arXiv:1802.02549`; *V. S. Braunack-Meyer*, loc. cit.].

In §7 $N_{\mathrm{dg}}\mathrm{Mod}_{\Omega C}^{\tau}$ is used to compute an explicit model for the colimit of an ∞-local system as a twisted tensor product between the dg coalgebra C of chains on the space and the dg ΩC-modules determined by the ∞-local system. An immediate consequence of the main results in the paper is an alternative and more conceptual proof of Brown's classical theorem on modelling the chains on the total space of a fibration in terms of chains on the base and chains on the fiber, as is discussed in §8.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18G35	Chain complexes (category-theoretic aspects), dg categories
18A05	Definitions and generalizations in theory of categories
18D20	Enriched categories (over closed or monoidal categories)
18M60	Operads (general)
18M75	Topological and simplicial operads
18N60	$(\infty,1)$-categories (quasi-categories, Segal spaces, etc.); ∞-topoi, stable ∞-categories
55P48	Loop space machines and operads in algebraic topology

Keywords:

quasi-categories; local systems; colimit; twisted tensor product

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