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Geometric multivector analysis. From Grassmann to Dirac. (English) Zbl 07139515

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This book addresses mathematics originating with Hermann Günther Grassmann [*H. G. Graßmann*, A new branch of mathematics. The 'Ausdehnungslehre' of 1844, and other works. Transl. from the German and with a note by Lloyd C. Kannenberg. With a foreword by Albert C. Lewis. Chicago, IL: Open Court Publishing (1995; Zbl 0923.01019)] and the work of William Kington Clifford in 1878 [`https://www.jstor.org/stable/pdf/2369379.pdf`]. The main part of the book is divided into two parts, namely, the first part (Chapters 2–5) dealing with basics of multivector algebra and the second part (Chapters 7–12) dealing with mulivector analysis. Chapter 1 ia a quick review of linear algebra, and Chapter 6 is an interlude for some preliminary material from analysis to be used in the remaining chapters.

Chapter 2 is concerned with Grassmann algebras. §2.2 is engaged in the geometric interpretation of k-vectors, the main result being the fundamental correspondence between simple k-vectors and k-dimensional subspaces in V (Proposition 2.2.3). The main result of §2.3 is geometry of Cramer's rule (Proposition 2.3.6). §2.4 gives composite k-vectors certain geometric meaning as oriented measures of curved k-surfaces in an affine space X. §2.5 deals with multicovectors. §2.6 is concerned with interior products and Hodge stars. The main result of §2.7 is geometry of cofactor formulas (Proposition 2.7.1). The main result of §2.8 is anticommutation relation between exterior and interior products (Theorem 2.8.1). The main result of §2.9 is Plücker relations (Theorem 2.9.1).

Chapter 3 is concerned with Clifford algebras, combining the exterior and interior products to obtain a *hypercomplex* associative product on $\wedge V$ in §3.1. §3.2 shows how the Clifford algebra reduces to Hamilton's classical algebra of quaternions. §3.3 addresses existence and uniqueness of Clifford algebras as well as their universality. §3.4 discusses that the best way to view Clifford algebra is rather as an algebra of matrices from a geometric viewpoint, the Clifford product corresponding to matrix multiplication.

Chapter 4 is concerned with rotations and Möbius maps. The main result of §4.1 is Cartan-Dieudonné theorem (Theorem 4.1.3) claiming that every isometry is a composition of hyperplane reflections. In classical complex analysis, multiplication by the exponential

$$z \mapsto e^{i\phi} z$$

gives rotation counterclockwise through an angle ϕ. §4.2 shows how the argument $i\phi$ of the exponential function generalizes to bivectors $\Delta^2 V$. Consider the smooth map

$$\exp : \underline{SO}(V) \to SO(V)$$

from the linear tangent space of skew-symmetric maps to the smooth manifold of isometries. The global behavior of exp is considered in §4.3 and §4.4. The main result of §4.5 is Theorem 4.5.12 claiming that

$$\text{fractional linear} = \text{Möbius} = \text{global conformal}$$

§4.6 establishes a double homomorphism taking spacetime isometries to Euclidean Möbius maps.

Chapter 5 is concerned with spinors in inner product spaces. §5.1 gives the standard representation as the most crucial example of a spinor space (Example 5.1.5). The main result of §5.2 is Theorem 5.2.3 claiming the uniqueness of minimal representations, which is indispensable for the geometric construction of spinors. The main result of §5.3 is Proposition 5.3.5 claiming uniqueness of spinor maps. The principal objective in §5.4 is to determine all possible abstract complex spinor and tensor spaces of three and four-dimensional Euclidean space, up to isomorphism (Proposition 5.4.2 and Proposition 5.4.8).

Chapter 7 is concerned with multivector calculus. Every mathematical student knows well the fundamental theorem of calculus

$$\int_a^b f'(x) = f(b) - f(a)$$

as well as its generalization to dimensions two and three, namely, the Green, Gauss and Stokes theorems. §7.3 establishes the general Stokes theorem, from which all integral theorems of the above kind in affine space follow. §7.6 discusses how Hodge decompositions work and how they can be used to solve boundary value problems (BVPs), which is followed by Chapter 10 through a more thorough study of Hodge decompositions. The highlight of §7.2 is commutation of pullback and exterior derivative (Theorem 7.2.9), that of §7.4 is Cartan's magic formula (Theorem 7.4.7), and that of §7.5 is the Poincaré theorem (Theorem 7.5.2).

Chapter 8 is concerned with hypercomplex analysis, a higher-dimensional generalization of classical complex analysis to Euclidean spaces in general, in which Cauchy-Riemann equations are to be replaced by a Dirac equation

$$\nabla_\triangle F(x) = 0$$

using the nabla operator induced by the Clifford product. The higher-dimensional complex analysis obtained from the Dirac equation and Clifford algebra has been developed since the 1980s, pioneered by [*F. Brackx* et al., Clifford analysis. Boston - London - Melbourne: Pitman Advanced Publishing Program (1982; Zbl 0529.30001)] under the name of *Clifford analysis*. Based on [*S. Axler* et al., Harmonic function theory. New York: Springer-Verlag (1992; Zbl 0765.31001); Harmonic function theory. 2nd ed. New York, NY: Springer (2001; Zbl 0959.31001)], §8.2 gives power series expansions in higher dimensions. §8.3 addresses splitting into Hardy subspaces. Theorem 8.3.2 is due to [*R. R. Coifman* et al., Ann. Math. (2) 116, 361–387 (1982; Zbl 0497.42012)], concerned with general Lipschitz graphs in the complex plane and equivalent to the L_2 boundedness of the Riesz transforms on Lipschitz surfaces. It has a much deeper extension, known as the Kato square root problem, which was finally settled affirmatively in [*P. Auscher* et al., Ann. Math. (2) 156, No. 2, 633–654 (2002; Zbl 1128.35316)].

Chapter 9 is concerned with Dirac wave equations. §9.1 deals with wave and spin equations. §9.2 describes briefly how Dirac equations appear in electromagnetic theory and quantum mechanics, showing geometrically that the electromagnetic field is a multivector field and that the Maxwell equations, written in terms of Clifford algebra, is no other than a Dirac wave equation. §9.3–§9.7 develop a theory for BVPs for Dirac equations, which is applied to electromagnetic scattering. §9.5 formulates integral equations for solving scattering problems for Dirac equations. §9.6 and §9.7 apply obtained good integral equations on good function spaces to scattering problems for electromagnetic waves.

Following §7.6, Chapter 10 is devoted to Hodge decompositions. In the anaysis of Hodge decompositions on domains, the regularity and curvature of the boundary play a significant role through Weitzenböck formulas. Hodge decompositions can also be considered on manifolds, for which the curvature of the manifold in the interior of the domain enters the picture. This is a central idea in Chapters 11 and 12. The discussion in Chapter 10 is limited to domains in affine space lest one confronts the technicalities of vector bundles. As in §§9.3–9.6, the author's basic philosophy in Chapter 10 is that one should handle Hodge decompositions by first-order operators, meaning concretely that one should study Hodge decompositions as far as possible by using Γ, Γ^* and Π rather than involving the abstract Laplace operator Π^2. §10.1, based on the survey paper [*A. Axelsson* and *A. McIntosh*, in: Advances in analysis and geometry. New developments using Clifford algebras. Based on the satellite conference to the ICM 2002 in Beijing, Macau, China, August 15–18, 2002. Basel: Birkhäuser. 3–29 (2004; Zbl 1174.58300)], shows how Hilbert space operators Γ with the property $\Gamma^2 = 0$ naturally induce splittings of the function space, generalizing Hodge decompositions. From §10.2 the author studies the nilpotent operators d and δ on bounded domains D, at least Lipschitz regular, in Euclidean space X. The main idea in §10.2 is to use the commutation theorem (Theorem 7.2.9) so as to reduce the problems to smooth domains. The main result in §10.3 is Theorem 10.3.1, concerned with Hodge decompositions on Lipschitz domains. §10.4 is concerned with Bogovskij and Poincaré potentials. §10.5 deals with Čech cohomology, while §10.6 is engaged in de Rham cohomology.

Base on the author's graduate course in Gothenburg 2014, Chapters 11 and 12 consider general compact Riemannian manifolds M, focusing on global smooth analysis in place of local nonsmooth analysis. Chapter 11 is concerned with multivector and spinor bundles. It is rather straightforward to construct a multivector bundle after Chapter 7, which is fulfilled in §11.2. In §11.6 the author uses Čech $\mathbb{Z}_2$ cohomology to investigate when there are topological obstructions for spinor bundles to exist, and if they exist, how many different spinor bundles there are over M. §11.3, §11.5 and §11.6 deal with Weitzenböck identities in preparation for Chapter 12. §11.4 addresses Liouville's theorem on conformal maps after [*H. Flanders*, J. Math. Mech. 15, 157–161 (1966; Zbl 0141.27103)].

Chapter 12 studies two celebrated generalizations of the Gauss-Bonnet theorem, namely, the Chern-Gauss-Bonnet theorem [*S.-s. Chern*, Ann. Math. (2) 46, 674–684 (1945; Zbl 0060.38104)] and the Atiyah-Singer index theorem [*M. F. Atiyah* and *I. M. Singer*, Bull. Am. Math. Soc. 69, 422–433 (1963; Zbl 0118.31203)]. For both proofs the author uses the well-known heat equation method. The proofs of the index theorems follow the book [*Y. Yu*, The index theorem and the heat equation method. Singapore: World Scientific (2001; Zbl 0986.58013)], which does not rely on pseudodifferential operators. §12.1 and §12.2 contain material on L_2 Dirac operators and charts, preliminary to the proofs of the index theorems. §12.3 is devoted to the Chern-Gauss-Bonnet theorem, and §12.4 deals with Atiyah-Singer index theorem.

All in all, the book is carefully prepared and well presented, and I recommend the book as well as [*J. P. Fortney*, A visual introduction to differential forms and calculus on manifolds. Cham: Birkhäuser (2018; Zbl 1419.58001)] for students who have just mastered vector calculus.

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MSC:

58-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to global analysis
53-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to differential geometry
15-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to linear algebra

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