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Homotopical algebra for Lie algebroids. **(English summary)**
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The principal objective in this paper is to give a model-categorical description of the homotopy theory of differential graded Lie algebroids over a commutative dg-algebra of characteristic 0 for the sake of using it to study the role of dg-Lie algebroids in deformation theory.

J. Lurie ["Derived algebraic geometry X: formal moduli problems", Harvard Univ., 2011; per bibliography] and J. P. Pridham [Adv. Math. **224** (2010), no. 3, 772–826; MR2628795] have established an equivalence between the homotopy theory of dg-Lie algebras and a certain homotopy theory of formal moduli problems over k. One can extend the idea to more general commutative dg-algebras A of characteristic 0 by describing a formal neighborhood of Spec(A) inside a moduli space X in terms of a dg-Lie algebroid over A.

Indeed, D. Gaitsgory and N. Rozenblyum [*A study in derived algebraic geometry. Vol. I. Correspondences and duality*, Math. Surveys Monogr., 221, Amer. Math. Soc., Providence, RI, 2017; MR3701352; *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*, Math. Surveys Monogr., 221, Amer. Math. Soc., Providence, RI, 2017; MR3701353] essentially defined Lie algebroids to be formal moduli problems over Spec(A) and developed their theory in these terms. This paper is, in a sense, a complement to Gaitsgory and Rozenblyum's work, providing a rigid, point-set model for the homotopy theory of Lie algebroids in terms of the dg-version of the familiar notion of a Lie algebroid [G. S. Rinehart, Trans. Amer. Math. Soc. **108** (1963), 195–222; MR0154906].

The author's proof is based upon an analysis of pushouts of generating trivial cofibrations of dg-Lie algebroids, proceeding along the same lines as for algebras over operads with the exception that the pushout of a generating trivial cofibration is not necessarily an injection. The author of the present paper showed in [Adv. Math. **354** (2019), 106750; MR3989531] that the homotopy theory of dg-Lie algebroids over cofibrant A provided by the above result is equivalent to the homotopy theory of formal moduli problems over A, meaning that dg-Lie algebroids can indeed be exploited as algebraic models for the formal neighborhood of Spec(A) inside moduli spaces. *Hirokazu Nishimura*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.