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Cell 2-representations and categorification at prime roots of unity. (English) Zbl 07152659

Adv. Math. 361, Article ID 106937, 66 p. (2020).

If q is a root of unity, then the quantum group $U_q(\mathfrak{g})$ associated to a finite-dimensional Lie algebra has a finite-dimensional quotient $\mathfrak{u}_q(\mathfrak{g})$ which has been used to construct invariants of 3-dimensional manifolds forming a 3-dimensional topological field theory, as was discussed in [*N. Yu. Reshetikhin* and *V. G. Turaev*, Commun. Math. Phys. 127, No. 1, 1–26 (1990; Zbl 0768.57003)]. These invariants have been linked to the Jones polynomials by *E. Witten* [Adv. Ser. Math. Phys. 9, 239–329 (1989; Zbl 0726.57010); ibid. 17, 361–451 (1994; Zbl 0818.57014)]. *L. Crane* and *I. B. Frenkel* [J. Math. Phys. 35, No. 10, 5136–5154 (1994; Zbl 0892.57014)] suggested to replace such Hopf algebras as $\mathfrak{u}_q(\mathfrak{g})$ by categories in the pursuit of constructing 4-dimensional topological quantum field theories (TQFT) by algebraic means.

A categorification of quantum groups at roots of unity is needed in order to make 4-dimensional TQFT a reality. An important idea was introduced in [*M. Khovanov*, J. Knot Theory Ramifications 25, No. 3, Article ID 1640006, 26 p. (2016; Zbl 1370.18017)] by observing that working algebra objects in the category of modules over a finite-dimensional Hopf algebra H gives a way of categorifying algebras over the Grothendieck ring of the stable category of H-modules. Khovanov's idea was subsequently further developed in [*Y. Qi*, Compos. Math. 150, No. 1, 1–45 (2014; Zbl 1343.16010)], where p-dg algebras and their modules are formally introduced in analogy to the theory of dg modules over dg algebras generalizing complexes of modules over an algebra. It was in [*M. Khovanov* and *Y. Qi*, Quantum Topol. 6, No. 2, 185–311 (2015; Zbl 1352.81038)] and [*B. Elias* and *Y. Qi*, Adv. Math. 288, 81–151 (2016; Zbl 1329.81238); ibid. 299, 863–930 (2016; Zbl 1355.81096)] that milestones in realizing the program of categorifying quantum groups at roots of unity were established.

The principal objective in this paper consisting of seven sections is to study p-dg enriched 2-categories and to introduce cell 2-representations for a class of such structures obeying finiteness conditions. The authors investigate their basic properties and relation to additive cell 2-representations. Structural results about passing to stable 2-representations which are compatible with the triangulated structure of the stable 2-categories are presented. Now is a brief synopsis in order.

§2 introduces all technical results on the level of 1-categories enriched with p-differentials, and investigates their compact semi-free modules, which are the suitable analogue of free modules in this context. §3 gathers preliminary results about the kind of p-dg 2-categories. Their p-dg 2-representations are studied in §4. §5 and §6 are the core of the paper. §5 introduces cell 2-representations for strongly finitary p-dg 2-categories. In [*V. Mazorchuk* and *V. Miemietz*, Int. Math. Res. Not. 2016, No. 24, 7471–7498 (2016; Zbl 1404.18014)] it was shown that any fiat 2-category that is simple in a suitable sense is biequivalent to a certain 2-category $\mathcal{C}_A$ constructed from projective bimodules over a finite-dimensional algebra A. §6 gives the construction of p-dg analogues $\mathcal{C}_{\mathcal{A}}$ of such 2-categories associated to a p-dg category $\mathcal{A}$. §7 applies some results to the cyclotomic quotient $\mathcal{U}_\lambda$ of the p-dg 2-category $\mathcal{U}$ introduced in [loc. cit., Zbl 1329.81238] to categorify the small quantum group associated to $\mathfrak{sl}_2$.

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MSC:

18D05	Double categories, 2-categories, bicategories and generalizations (MSC2010)
18D20	Enriched categories (over closed or monoidal categories)
17B10	Representations of Lie algebras and Lie superalgebras, algebraic theory (weights)

Keywords:

2-representation theory; enriched 2-categories; categorification at roots of unity; hopfological algebra

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