An index formula for the relative Hodge-Kodaira theory

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The Purpose of Talk

The purpose of this talk is to prove an index formula for the relative de Rham cohomology groups and is to give an interpretation of the index formula in terms of harmonic integrals. In deriving our index formula, the theory of polyharmonic forms satisfying an interior boundary condition plays a fundamental role.

Bird's-Eye View

Algebraic Topology	Differential Geometry	Partial Differential Equations
Simplicial Complex	Compact Manifold	Laplace-Beltrami Operator
Simplex	Differential Form	Current (distribution-valued differential form)
Simplicial Cohomology Group	de Rham Cohomology	Hodge-Kodaira Decomposition
Euler-Poincare Characteristic	Euler-Poincare Characteristic	Analytical Index

William Valance Douglas Hodge

William Valance Douglas Hodge (1903-1975)
British mathematician

Georges de Rham

Georges de Rham (1903-1990)
Swiss mathematician

Kunihiko Kodaira

Kunihiko Kodaira (1915-1997)
Japanese mathematician

Relative de Rham complex

- X compact manifold without boundary of dimension n
- Y compact submanifold of X of dimension m
- $\iota : Y \rightarrow X$ (natural inclusion map)

Interior Boundary Value Problem

$$\begin{cases} (I+\Delta)(d+\delta)\alpha = S\otimes\delta_Y & \text{on } X \\ i^*\alpha = 0 & \text{on } Y \end{cases}$$

Operators

- d **exterior derivative**
- δ **codifferential**
- $\Delta = d\delta + \delta d$ **Laplace-Beltrami operator**
- δ_Y **Dirac delta function supported on** Y
- ι^* **Pull-back of the natural inclusion** ι

Concrete Examples

Example 1

$$\dim X = 4, \quad \dim Y = 2$$

$$D = \begin{pmatrix} d + \delta & -(I + \Delta)^{-1}\left(\bullet \otimes \delta_Y\right) \\ i^* & 0 \end{pmatrix}$$

Example 1

$$\operatorname{ind} D = \chi(X) - \chi(Y)$$

$$= \frac{1}{32\pi^2}\int_X \left(\kappa^2 - 4\|\mathrm{Ric}\|^2 + \|R\|^2\right)\mu_X - \frac{1}{2\pi}\int_Y K\,\mu_Y$$

Curvatures

- $R = R_{ikj\ell}$ **Riemannian curvature tensor**
- $\mathrm{Ric} = R_{ikjk}$ **Ricci curvature**
- $\kappa = R_{ijij}$ **Scalar curvature**
- K **Gaussian curvature**

Example 2

$$\dim X = 6, \quad \dim Y = 4$$

$$D = \begin{pmatrix} d + \delta & -(I + \Delta)^{-1}\left(\bullet \otimes \delta_Y\right) \\ i^* & 0 \end{pmatrix}$$

Example 2

$$\operatorname{ind} D = \chi(X) - \chi(Y)$$

$$= \frac{1}{384\pi^3}\int_X (\kappa^3 - 12\kappa\|\mathrm{Ric}\|^2 + 3\kappa\|R\|^2 + 16 R^i_j R^j_k R^k_i$$

$$+24 R^{ij} R^{k\ell} R_{ikj\ell} - 24 R^{ij} R_{iabc} R_j^{abc} - 8 R^{ijk\ell} R^{rs}_{ik} R_{js\ell r} + 2 R^{ijk\ell} R_{ijrs} R^{rs}_{k\ell})\mu_X$$

$$-\frac{1}{32\pi^2}\int_Y \left(\kappa_Y^2 - 4\|\mathrm{Ric}_{\mathrm{Y}}\|^2 + \|R_Y\|^2\right)\mu_Y$$

References

- **S.- S. Chern**:

A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds,

Ann. of Math. (2) 45, 747-752 (1944)

Main Result

Our result may be stated as follows:

Brownian motion describes the topology of a compact Riemannian manifold through its Euler-Poincare characteristic.

The de Rham Complex
for
Manifolds without Boundary

References

John Roe: Elliptic operators, topology and asymptotic methods, 2nd edition, Pitman Research Notes in Mathematics Series, No. 395, Longman, Harlow, 1998.

H.B. Lawson, Jr. and M.L. Michelsohn: Spin geometry, second printing, Princeton Mathematical Series, No. 38, Princeton University Press, Princeton, New Jersey, 1994.

Part I

Hodge-Kodaira Theorem

de Rham complex (1)

- M : n-dimensional, Riemannian manifold **without** boundary
- $\Omega^k(M)$ (differential forms of degree k)
- $\Omega^\cdot(M) = \oplus_{k=0}^n \Omega^k(M)$
- d exterior derivative
- δ codifferential

de Rham complex (2)

$$\Omega^{k-1}(M) \xrightarrow{d^{k-1}} \Omega^{k}(M) \xrightarrow{d^{k}} \Omega^{k+1}(M)$$

$$d^k\, d^{k-1} = 0$$

de Rham complex (3)

- $Z^k(M) = \left\{ \alpha \in \Omega^k(M) : d\alpha = 0 \right\} = \mathrm{Ker}\, d^k$ **(closed forms)**
- $B^k(M) = \left\{ d\beta \in \Omega^k(M) : \beta \in \Omega^{k-1}(M) \right\} = \mathrm{Im}\, d^{k-1}$ **(exact forms)**
- $H^k(M) = \mathrm{Ker}\, d^k / \mathrm{Im}\, d^{k-1} = Z^k(M) / B^k(M)$ **(de Rham cohomology group)**

de Rham Theorem

- $H^k(M;\mathbf{R})$

  (simplicial cohomology group)

- $b_k(M) = \dim H^k(M;\mathbf{R})$

  (Betti number)

- $\boxed{H^k(M) = Z^k(M)/B^k(M) \;\cong H^k(M;\mathbf{R})}$

  (de Rham's theorem)

- $\chi(M) = \sum_{k=0}^{n}(-1)^k \dim H^k(M;\mathbf{R}) = \sum_{k=0}^{n}(-1)^k \dim H^k(M)$

  (Euler - Poincare characteristic)

Hodge and Kodaira Theorem

- $\Delta = (d+\delta)^2 = d\delta + \delta d$

  (Laplace - Beltrami operator)

- $\mathsf{H}^k(M) = \mathrm{Ker}^k\Delta = \left\{\alpha \in \Omega^k(M) : \Delta\alpha = 0\right\} = \mathrm{Ker}^k d \cap \mathrm{Ker}^k \delta$

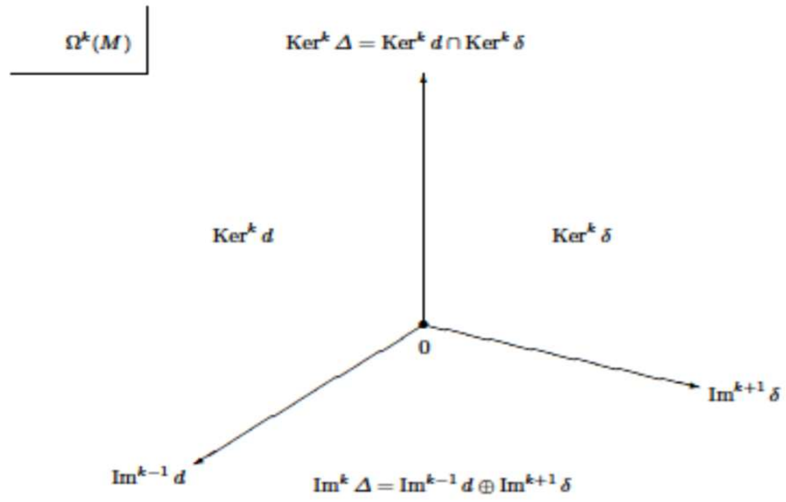
  (harmonic forms)

- $\boxed{H^k(M) \cong \mathsf{H}^k(M)}$ **(Hodge - Kodaira theorem)**

- $\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M) = \sum_{k=0}^{n} (-1)^k \dim \mathsf{H}^k(M)$

  (Euler - Poincare characteristic)

Hodge and Kodaira decomposition



Clifford Algebra and Dirac Operators

Clifford Algebra

- M :

n-dimensional, Riemannian manifold **without** boundary

- $V = T_m^*(M)$: **real inner product space**

$< e^1, e^2, \cdots, e^n >$ **orthonormal basis of** V

- $V^* = T_m(M)$:

$< e_1, e_2, \cdots, e_n >$ **orthonormal (dual) basis of** V^*

Clifford Bundle (1)

- $C\ell(V) = \Lambda^{\bullet} T_m^*(M)$:

Bundle of Clifford Algebras

- $S = C\ell(M) = \Lambda^{\bullet} T^*(M) = \bigcup_{m \in M} \Lambda^{\bullet} T_m^*(M)$:

Clifford Bundle

Euler Grading Operator

- $\varepsilon = (-1)^q \quad \text{on } \Lambda^q T^*(M)$:

Euler grading operator

$\varepsilon^2 = 1 \quad \text{on } \Lambda^q T^*(M)$

- $C\ell_+(M) = (1+\varepsilon)\Lambda^* T^*(M) = \Lambda^{\text{even}} T^*(M)$
- $C\ell_-(M) = (1-\varepsilon)\Lambda^* T^*(M) = \Lambda^{\text{odd}} T^*(M)$

$\Rightarrow$

$$\boxed{C\ell(M) = C\ell_+(M) \oplus C\ell_-(M)}$$

Clifford Bundle (2)

$$\varepsilon = (-1)^q \quad \text{on } \Lambda^q T^*(M)$$

$\Rightarrow$

- $S_+ = C\ell_+(M) = \Lambda^{\text{even}} T^*(M)$:

$+1$ **eigenspace of** ε

- $S_- = C\ell_-(M) = \Lambda^{\text{odd}} T^*(M)$:

-1 **eigenspace of** ε

Sections of Clifford Bundle

- $C^{\infty}(S) = C^{\infty}\left(\Lambda^{\bullet} T^{*}(M)\right)$ **differential forms**
- $C^{\infty}(S_{+}) = C^{\infty}\left(\Lambda^{\text{even}} T^{*}(M)\right)$ **differential forms of even degree**
- $C^{\infty}(S_{-}) = C^{\infty}\left(\Lambda^{\text{odd}} T^{*}(M)\right)$ **differential forms of odd degree**

$\Rightarrow$

$$\boxed{C^{\infty}(S) = C^{\infty}(S_{+}) \oplus C^{\infty}(S_{-})}$$

Dirac Operators (1)

- $\nabla : C^\infty\left(\Lambda^\bullet T^*(M)\right) \to T^*(M) \otimes C^\infty\left(\Lambda^\bullet T^*(M)\right)$: **Levi - Civita connection**
- $c : C^\infty\left(T^*(M) \otimes \Lambda^\bullet T^*(M)\right) \to C^\infty\left(\Lambda^\bullet T^*(M)\right)$:

$\boxed{c(e)\omega = e \wedge \omega - \iota(e)\omega}$ **Clifford action**

- $D : C^\infty\left(\Lambda^\bullet T^*(M)\right) \overset{\nabla}{\to} C^\infty\left(T^*(M) \otimes \Lambda^\bullet T^*(M)\right) \overset{c}{\to} C^\infty\left(\Lambda^\bullet T^\bullet(M)\right)$

$\boxed{D = c \circ \nabla}$ **Dirac operator**

Exterior Derivative and Codifferential

- $d\omega = \sum_{j=1}^{n} e^{j} \wedge \nabla_{e_j} \omega$ **exterior derivative**
- $\delta\omega = -\sum_{j=1}^{n} \iota\left(e^{j}\right) \nabla_{e_j} \omega$ **codifferential**

Dirac Operators (2)

- $D\omega := \sum_{j=1}^{n} e\left(e^j\right)\nabla_{e_j}\omega = \sum_{j=1}^{n} e^j \wedge \nabla_{e_j}\omega - \sum_{j=1}^{n} \iota\left(e^j\right)\nabla_{e_j}\omega$

  $= d\omega + \delta\omega$ **Dirac operator**

$\Rightarrow$

$$D = d + \delta : C^\infty\left(\Lambda^\bullet T^*(M)\right) \to C^\infty\left(\Lambda^\bullet T^*(M)\right)$$

Sections of Clifford Bundle

- $C^{\infty}\left(\Lambda^{\bullet}T^{*}(M)\right)=C^{\infty}\left(\Lambda^{\text{even}}T^{*}(M)\right)\oplus C^{\infty}\left(\Lambda^{\text{odd}}T^{*}(M)\right)$
- $C^{\infty}(S)=C^{\infty}(S_{+})\oplus C^{\infty}(S_{-})$

Graded Dirac Operators

- $D_+ : C^\infty(S_+) \xrightarrow{D} C^\infty(S_-)$
- $D_- : C^\infty(S_-) \xrightarrow{D} C^\infty(S_+)$

$\Rightarrow$

- $D_+ = (d+\delta)_+ : C^\infty\left(\Lambda^{\text{even}} T^*(M)\right) \to C^\infty\left(\Lambda^{\text{odd}} T^*(M)\right)$
- $D_- = (d+\delta)_- : C^\infty\left(\Lambda^{\text{odd}} T^*(M)\right) \to C^\infty\left(\Lambda^{\text{even}} T^*(M)\right)$

(Euler characteristic operators)

The fundamental property of Dirac operator

$$D^2 = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} D_- D_+ & 0 \\ 0 & D_+ D_- \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$$

Index of a Graded Dirac Operator (1)

$$\mathrm{ind}\, D := \dim \mathrm{Ker}\, D_+ - \dim \mathrm{Ker}\, D_-$$

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} : \begin{matrix} C^\infty(S_+) \\ \oplus \\ C^\infty(S_-) \end{matrix} \rightarrow \begin{matrix} C^\infty(S_+) \\ \oplus \\ C^\infty(S_-) \end{matrix}$$

Index of a Graded Dirac Operator (2)

$$\mathrm{Ker}\, D_+ \simeq \oplus H^{2k}(M) \quad \textbf{(harmonic forms of even degree)}$$

$$\mathrm{Ker}\, D_- \simeq \oplus H^{2k+1}(M) \quad \textbf{(harmonic forms of odd degree)}$$

Index formula for the Euler characteristic operator

(Hodge-Kodaira theorem)

$$D_+ = (d+\delta)_+ : C^\infty\left(\Lambda^{\text{even}} T^*(M)\right) \to C^\infty\left(\Lambda^{\text{odd}} T^*(M)\right)$$

$$D_- = (d+\delta)_- : C^\infty\left(\Lambda^{\text{odd}} T^*(M)\right) \to C^\infty\left(\Lambda^{\text{even}} T^*(M)\right)$$

$$\begin{aligned}
\operatorname{ind} D &= \dim \operatorname{Ker}(d+\delta)_+ - \dim \operatorname{Ker}(d+\delta)_- \\
&= \sum_{j=0}^{n} (-1)^j \dim H^j(M) \\
&= \sum_{j=0}^{n} (-1)^j \dim H^j(M;\mathbf{R}) \\
&= \chi(M)
\end{aligned}$$

Part II

Hirzebruch Signature Theorem

Clifford Algebra

• M : $\boxed{\dim M = 4k}$

$4k$**-dimensional, Riemannian manifold without boundary**

• $V = T_m^*(M)$:

$< e^1, e^2, \cdots, e^n >$ **orthonormal basis**

• $V^* = T_m(M)$:

$< e_1, e_2, \cdots, e_n >$ **orthonormal (dual) basis**

Clifford Bundle (1)

- $C\ell(V) = \Lambda^{\bullet} T_m^*(M)$:

Bundle of Clifford Algebras

- $S = C\ell(M) = \Lambda^{\bullet} T^*(M) = \bigcup_{m \in M} \Lambda^{\bullet} T_m^*(M)$:

Clifford Bundle

Grading Operator (1)

- $\tau = (-1)^{k+q(q-1)/2} *$ on $\Lambda^q T^*(M)$:

Grading operator

$\Rightarrow$

$\tau^2 = 1$ on $\Lambda^q T^*(M)$

$\tau = *$ on $\Lambda^{2k} T^*(M)$ (**Hodge star operator**)

Grading Operator (2)

- $\tau = (-1)^{k+q(q-1)/2} *$ on $\Lambda^q T^*(M)$

Grading operator

- $C\ell_+(M) = (1+\tau)\Lambda^\bullet T^*(M)$
- $C\ell_-(M) = (1-\tau)\Lambda^\bullet T^*(M)$

$\Rightarrow$

$$\boxed{C\ell(M) = C\ell_+(M) \oplus C\ell_-(M)}$$

Clliford Bundle (2)

- $\tau = (-1)^{k+q(q-1)/2} *$ on $\Lambda^q T^*(M)$
- $C\ell_+(M) = (1+\tau)\Lambda^\bullet T^*(M)$:

$+1$ **eigenspace of** τ

- $C\ell_-(M) = (1-\tau)\Lambda^\bullet T^*(M)$:

-1 **eigenspace of** τ

Sections of Clifford Bundle

- $C^{\infty}(S) = C^{\infty}\left(\Lambda^{\bullet}T^{*}(M)\right)$
- $C^{\infty}(S_{+}) = C^{\infty}\left((1+\tau)\Lambda^{\bullet}T^{*}(M)\right)$
- $C^{\infty}(S_{-}) = C^{\infty}\left((1-\tau)\Lambda^{\bullet}T^{*}(M)\right)$

$\Rightarrow$

$$\boxed{C^{\infty}(S) = C^{\infty}(S_{+}) \oplus C^{\infty}(S_{-})}$$

Dirac Operators (1)

- $\nabla : C^\infty\left(\Lambda^* T^*(M)\right) \to T^*(M) \otimes C^\infty\left(\Lambda^* T^*(M)\right)$: **Levi - Civita connection**
- $c : C^\infty\left(T^*(M) \otimes \Lambda^* T^*(M)\right) \to C^\infty\left(\Lambda^* T^*(M)\right)$: $\boxed{c(e)\omega = e \wedge \omega - \iota(e)\omega}$

Clifford action

- $D : C^\infty\left(\Lambda^* T^*(M)\right) \overset{\nabla}{\to} C^\infty\left(T^*(M) \otimes \Lambda^* T^*(M)\right) \overset{c}{\to} C^\infty\left(\Lambda^* T^*(M)\right)$: $\boxed{D = c \circ \nabla}$

Dirac operator

Exterior Derivative and Codifferential

- $d\omega = \sum_{j=1}^{n} e^{j} \wedge \nabla_{e_j}\omega$ **exterior derivative**
- $\delta\omega = -\sum_{j=1}^{n} \iota\left(e^{j}\right)\nabla_{e_j}\omega$ **codifferential**

Dirac Operators (2)

$$\bullet\, D\omega := \sum_{j=1}^{n} e\left(e^j\right)\nabla_{e_j}\omega = \sum_{j=1}^{n} e^j \wedge \nabla_{e_j}\omega - \sum_{j=1}^{n} \iota\left(e^j\right)\nabla_{e_j}\omega$$

$$= d\omega + \delta\omega$$

$$\Rightarrow$$

$$D = d + \delta : C^{\infty}\left(\Lambda^{\bullet}T^{*}(M)\right) \to C^{\infty}\left(\Lambda^{\bullet}T^{*}(M)\right)$$

Graded Dirac Operators (1)

$$D = d + \delta$$
$$D\tau = -\tau D$$

$\Rightarrow$

$$D_+ : C^\infty(S_+) \xrightarrow{D} C^\infty(S_-)$$
$$D_- : C^\infty(S_-) \xrightarrow{D} C^\infty(S_+)$$

(signature operator)

Index of a Graded Dirac Operator

$$\operatorname{ind} D = \dim \operatorname{Ker} D_+ - \dim \operatorname{Ker} D_-$$

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} : \begin{matrix} C^\infty(S_+) \\ \oplus \\ C^\infty(S_-) \end{matrix} \rightarrow \begin{matrix} C^\infty(S_+) \\ \oplus \\ C^\infty(S_-) \end{matrix}$$

Graded Dirac Operators (2)

$$C\ell(M) = C\ell_+(M) \oplus C\ell_-(M)$$

- $C\ell_+(M) = (1+\tau)\Lambda^\bullet T^*(M)$
- $C\ell_-(M) = (1-\tau)\Lambda^\bullet T^*(M)$

$\Rightarrow$

$$\ker D = \ker D_+ \oplus \ker D_-$$

$$\ker D_\pm = (1 \pm \tau)\ker D$$

Structure of Kernels (1)

$$\bullet \ker D = \mathsf{H} = \mathsf{H}^0 \oplus \cdots \oplus \mathsf{H}^{4k}$$

$$\Rightarrow$$

$$\tau = (-1)^{k+q(q-1)/2} * : \mathsf{H}^q \to \mathsf{H}^{4k-q}$$

isomorphism for $0 \leq q \leq 2k-1$

Structure of Kernels (2)

$$\bullet \mathrm{H}(q) := \mathrm{H}^{q} \oplus \mathrm{H}^{4k-q} = \mathrm{H}_{+}(q) \oplus \mathrm{H}_{-}(q)$$

$$\mathrm{H}_{\pm}(q) := (1 \pm \tau)\mathrm{H}(q)$$

$$\left(0 \le q \le 2k-1\right)$$

$$\mathrm{H}^{2k} = \mathrm{H}_{+}^{2k} \oplus \mathrm{H}_{-}^{2k} \quad \left(q = 2k\right)$$

Structure of Kernels (3)

$$\mathsf{H} = \mathsf{H}^{+} \oplus \mathsf{H}^{-}$$

$$= \left(\mathsf{H}^{+}(0) \oplus \cdots \oplus \mathsf{H}^{+}(2k-1) \oplus \boxed{\mathsf{H}_{+}^{2k}} \right)$$

$$\oplus \left(\mathsf{H}^{-}(0) \oplus \cdots \oplus \mathsf{H}^{-}(2k-1) \oplus \boxed{\mathsf{H}_{-}^{2k}} \right)$$

Structure of Kernels (4)

- $\ker D_{\pm} = (1 \pm \tau)\ker D = (1 \pm \tau)\mathsf{H}$

$\Rightarrow$

$$\ker D_{+} = (1 + \tau)\mathsf{H} = \mathsf{H}^{+}$$

$$\ker D_{-} = (1 - \tau)\mathsf{H} = \mathsf{H}^{-}$$

Index formula for the signature operator

(Hirzebruch signature theorem)

$$
\begin{aligned}
\operatorname{ind} D &= \dim \operatorname{Ker} D_+ - \dim \operatorname{Ker} D_- \\
&= \dim \mathsf{H}_+^{2k} - \dim \mathsf{H}_-^{2k} \\
&= \sigma(M^{4k}) \\
&= \int_M L_k(\Omega)_{4k}
\end{aligned}
$$

Hirzebruch *L*-Genus

- $L_1(\Omega)_4 = \dfrac{1}{3} p_1(\Omega)$
- $L_2(\Omega)_8 = \dfrac{1}{45}\left(7\, p_2(\Omega) - p_1(\Omega) \wedge p_1(\Omega)\right)$

Here : $p_i(\Omega)$ i-th **Pontrjagin form**

Part III

The de Rham Complex for Manifolds with Boundary

References

Peter B. Gilkey: Invariance theory, the heat equation, and the Atiyah-Singer index theorem. Second edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, Florida, 1995.

Clifford Algebra in the Interior

- M :

n-**dimensional, Riemannian manifold with boundary** ∂M

- $V = T_m^*(M)$:

$< e^1, e^2, \cdots, e^n >$ **orthonormal basis of** V

- $V^* = T_m(M)$:

$< e_1, e_2, \cdots, e_n >$ **orthonormal (dual) basis of** V^*

Clifford Bundle in the Interior

- $C\ell(V) = \Lambda^* T_m^*(M)$:

Bundle of Clifford Algebras

- $S = C\ell(M) = \Lambda^* T^*(M) = \bigcup_{m \in M} \Lambda^* T_m^*(M)$:

Clifford Bundle

Grading Operator in the Interior

- $\varepsilon = (-1)^q$ on $\Lambda^q T^*(M)$:

Euler grading operator

$\varepsilon^2 = 1$ on $\Lambda^q T^*(M)$

- $C\ell_+(M) = (1+\varepsilon)\Lambda^\bullet T^*(M) = \Lambda^{\text{even}} T^*(M)$
- $C\ell_-(M) = (1-\varepsilon)\Lambda^\bullet T^*(M) = \Lambda^{\text{odd}} T^*(M)$

$\Rightarrow$

$$\boxed{C\ell(M) = C\ell_+(M) \oplus C\ell_-(M)}$$

Clifford Algebra near the Boundary (1)

Near the boundary ∂M

- $< y^1, \cdots, y^{n-1}, a >$: **local coordinates**
- $< y^1, \cdots, y^{n-1} >$ **local coordinates for** ∂M
- $M = \{x : a(x) \geq 0\}$
- **The curves** $\{x(a) = (y_0, a) : a \in [0, \delta)\}$

unit speed geodesics perpendicular to ∂M

Clifford Algebra near the Boundary (2)

• $V = T_m^*(M)$:

$< e^1 = dy^1, \cdots, e^{n-1} = dy^{n-1}, e^n = da >$ **orthonormal basis**

$$T^*(M) \cong T^*(\mathbf{R}) \oplus T^*(\partial M)$$

• $V^* = T_m(M)$:

$< e_1, \cdots, e_{n-1}, e_n >$ **orthonormal (dual) basis**

Clifford Bundle near the Boundary

Near the boundary ∂M

- $\Omega(M) \simeq \Omega(\partial M) \oplus \left(da \wedge \Omega(\partial M)\right)$
- $\theta = \theta_t + da \wedge \theta_n$ **decomposition of differential forms**
- $\theta_t, \theta_n \in \Omega(\partial M)$ **tangential differential forms**

Grading Operator near the Boundary (1)

- $\theta = \theta_t + da \wedge \theta_n \in \Omega(\partial M) \oplus \left(da \wedge \Omega(\partial M)\right)$

Grading operator

$$\boxed{\alpha(\theta) = \theta_t - da \wedge \theta_n}$$

- $\alpha^2 = 1 \quad \text{on } \Lambda^q T^*(M)$

Grading Operator near the Boundary (2)

- $\theta = \theta_t + da \wedge \theta_n \in \Omega(\partial M) \oplus \left(da \wedge \Omega(\partial M)\right)$

$\alpha(\theta) = \theta_t - da \wedge \theta_n$

- $V_+(\alpha) = (1+\alpha)\Lambda^* T^*(M)$

$+1$ **eigenspace of** α

- $V_-(\alpha) = (1-\alpha)\Lambda^* T^*(M)$

-1 **eigenspace of** α

$$\Omega(\partial M) \oplus \left(da \wedge \Omega(\partial M)\right) \cong V_+(\alpha) \oplus V_-(\alpha)$$

Clifford Actions near the Boundary (1)

Clifford actions

- $c(e^j)\omega = e^j \wedge \omega - \iota(e^j)\omega, \quad 1 \le j \le n-1$

$c(e^j)$ **preserves the tangential differential forms** $\Omega(\partial M)$

- $\theta = \theta_t + da \wedge \theta_n \in \Omega(\partial M) \oplus \left(da \wedge \Omega(\partial M)\right)$

$$c(da)\left(\theta\right) = da \wedge \theta - \iota\left(da\right)\theta$$

$c(da)$ **inerchanges the factors of**

$\Omega(\partial M) \oplus \left(da \wedge \Omega(\partial M)\right) = V_+(\alpha) \oplus V_-(\alpha)$

Clifford Actions near the Boundary (2)

- $\theta = \theta_t + da \wedge \theta_n \in V_+(\alpha) \oplus V_-(\alpha)$

$$\boxed{c(da)\left(\theta\right) = da \wedge \theta_t - \theta_n}$$ **(Clifford Action)**

- $c(da) : V_-(\alpha) \to V_+(\alpha)$
- $c(da) : V_+(\alpha) \to V_-(\alpha)$

Relative and Absolute Boundary Conditions (1)

$$\theta = \theta_t + da \wedge \theta_n \in V_+(\alpha) \oplus V_-(\alpha)$$

$\iota : \partial M \to M$ **(natural inclusion map)**

- $B_r(\theta) = \frac{1}{2}(1+\alpha)\theta\big|_{\partial M} = \theta_t\big|_{\partial M} = \iota^*(\theta) \in V_+(\alpha)$

orthogonal projection on $V_+(\alpha)$

- $B_a(\theta) = \frac{1}{2}(1-\alpha)\theta\big|_{\partial M} = da \wedge \left(\theta_n\big|_{\partial M}\right) \in V_-(\alpha)$

orthogonal projection on $V_-(\alpha)$

Relative and Absolute Boundary Conditions (2)

$$\theta = \theta_t + da \wedge \theta_n \in V_+(\alpha) \oplus V_-(\alpha)$$

- $B_r(\theta) = \theta_t \big|_{\partial M} \in V_+(\alpha)$

$$\textbf{Ker } B_r = V_-(\alpha) \cong da \wedge \Omega\left(\partial M\right)$$

- $B_a(\theta) = da \wedge \left(\theta_n \big|_{\partial M}\right) \in V_-(\alpha)$

$$\textbf{Ker } B_a = V_+(\alpha) \cong \Omega\left(\partial M\right)$$

Exterior Derivative and Coddifferential near the Boundary (1)

- $$d\omega = \sum_{j=1}^{n-1} e^{j} \wedge \nabla_{e_j}\omega + da \wedge \nabla_{e_n}\omega$$
  $$= d'\omega + da \wedge \nabla_{e_n}\omega$$
- $$\delta\omega = -\sum_{j=1}^{n-1} \iota\left(e^{j}\right)\nabla_{e_j}\omega - \iota\left(da\right)\nabla_{e_n}\omega$$
  $$= \delta'\omega - \iota\left(da\right)\nabla_{e_n}\omega$$

Exterior Derivative and Coddifferential near the Boundary (2)

$$\theta = \theta_t + da \wedge \theta_n$$

- $d\theta = d'\theta_t + da \wedge d'\theta_n$
- $\delta\theta = \delta'\theta_t + da \wedge \left(\left(\delta'\theta_n \right)\Big|_{\partial M} \right)$

Boundary Conditions (4)

$$\theta = \theta_t + da \wedge \theta_n \in V_+(\alpha) \oplus V_-(\alpha)$$

- $$\boxed{B_r(d\theta)} = (d'\theta_t)\big|_{\partial M} = d'\left(\theta_t\big|_{\partial M}\right) = \boxed{d' B_r(\theta)}$$

- $$\boxed{B_a(\delta\theta)} = da \wedge \left((\delta'\theta_n)\big|_{\partial M}\right) = da \wedge \left(\delta'\left(\theta_n\big|_{\partial M}\right)\right) = \boxed{\delta' B_a(\theta)}$$

de Rham Complex with Boundary Condition (1)

$$\Omega_r(M) = \{\theta \in \Omega(M) : B_r(\theta) = 0\}$$

$$B_r(d\theta) = d' B_r(\theta)$$

$$\Omega_r^k(M) \xrightarrow{d^k} \Omega_r^{k+1}(M)$$

de Rham Complex with Boundary Condition (2)

$$\Omega_r^{k-1}(M) \xrightarrow{d^{k-1}} \Omega_r^{k}(M) \xrightarrow{d^{k}} \Omega_r^{k+1}(M)$$

$$d^k \, d^{k-1} = 0$$

Exterior Derivative with Boundary Condition

$$(a)\ D(d_r) = \Omega_r^k(M) = \left\{\theta \in \Omega^k(M) : B_r(\theta) = 0\right\}$$

$$(b)\ d_r\theta = d\theta, \quad \forall\, \theta \in D(d_r)$$

$$\Rightarrow$$

$$\boxed{\begin{array}{l} d_r^{\,*} = \delta \\ \delta^* = d_r \end{array}}$$

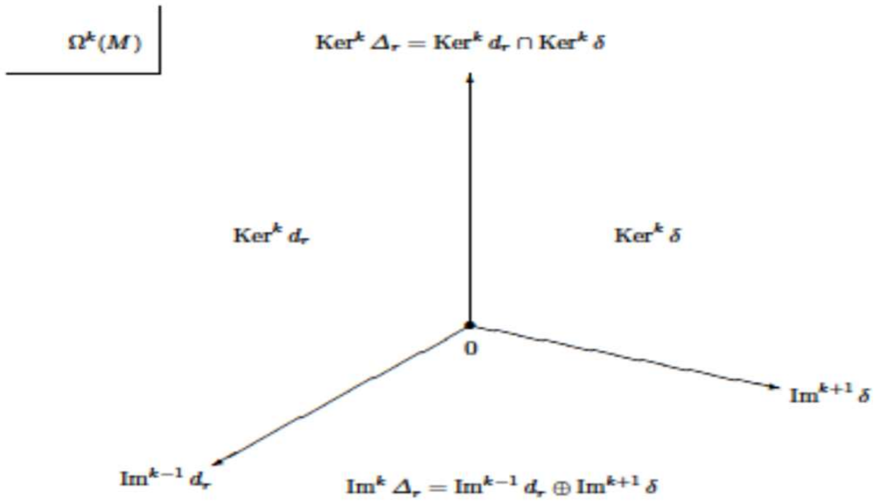
Green's Formula

$$\left(d\theta,\eta\right)=\left(\theta,\delta\eta\right)+\int_{\partial M}\left[B_r(\theta)\wedge B_r\left(*\eta\right)\right]_{n-1}$$

$$d_r{}^* = \delta$$

$$\delta^* = d_r$$

Hodge and Kodaira decomposition



Graded Dirac Operators

$$\bullet\; D_+ = (d_r + \delta)_+ : \Omega_r^{\text{even}}(M) \to \Omega_r^{\text{odd}}(M)$$

$$\bullet\; D_- = (d_r + \delta)_- : \Omega_r^{\text{odd}}(M) \to \Omega_r^{\text{even}}(M)$$

(Euler characteristic operators)

$$\Delta_r = \delta d_r + d_r \delta$$

The fundamental property of Dirac Operator

$$D^2 = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} D_- D_+ & 0 \\ 0 & D_+ D_- \end{pmatrix} = \begin{pmatrix} \Delta_r & 0 \\ 0 & \Delta_r \end{pmatrix}$$

$$\Delta_r = \delta d_r + d_r \delta$$

Index of a Graded Dirac Operator

$$\operatorname{ind} D = \dim \operatorname{Ker} D_+ - \dim \operatorname{Ker} D_-$$

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} : \begin{matrix} \Omega_r^{\text{even}}(M) \\ \oplus \\ \Omega_r^{\text{odd}}(M) \end{matrix} \to \begin{matrix} \Omega_r^{\text{even}}(M) \\ \oplus \\ \Omega_r^{\text{odd}}(M) \end{matrix}$$

Index formula for the de Rham Complex with Boundary Condition

$$D_+ = (d_r + \delta)_+ : \Omega_r^{\text{even}}(M) \to \Omega_r^{\text{odd}}(M)$$

$$D_- = (d_r + \delta)_- : \Omega_r^{\text{odd}}(M) \to \Omega_r^{\text{even}}(M)$$

$\Rightarrow$

$$\begin{aligned}\operatorname{ind}(d_r + \delta) &= \dim \operatorname{Ker}(d_r + \delta)_+ - \dim \operatorname{Ker}(d_r + \delta)_- \\ &= \sum_{j=0}^{m} (-1)^j \dim H^j(M, \partial M; \mathbf{R}) \\ &= \chi(M) - \chi(\partial M)\end{aligned}$$

Dual de Rham Complex with Boundary Condition

$$\Omega_a(M) = \{\theta \in \Omega(M) : B_a(\theta) = 0\}$$

$$B_a(\delta\theta) = \delta' B_a(\theta)$$

$$\Rightarrow$$

$$\Omega_a^{k-1}(M) \xleftarrow{\delta^k} \Omega_a^k(M) \xleftarrow{\delta^{k+1}} \Omega_a^{k+1}(M)$$

Codifferential with Boundary Condition

$$(a)\ D(\delta_a) = \Omega_a^k(M) = \left\{\eta \in \Omega^k(M) : B_a(\eta) = 0\right\}$$

$$(b)\ \delta_a \eta = \delta\eta, \quad \forall\, \eta \in D(\delta_a)$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} \delta_a{}^* &= d \\ d^* &= \delta_a \end{aligned}}$$

Boundary Conditions and Star Operators (1)

$$\theta = \theta_t + da \wedge \theta_n \in V_+(\alpha) \oplus V_-(\alpha)$$

- $B_r\left(*\left(da \wedge \theta_n\right)\right) = (-1)^{n-1} *'\left(\theta_n\big|_{\partial M}\right)$
- $B_a(*\theta_t) = *'\left(\theta_t\big|_{\partial M}\right) \wedge da \in V_-(\alpha)$

Boundary Conditions and Star Operators (2)

$$\theta = \theta_t + da \wedge \theta_n \in V_+(\alpha) \oplus V_-(\alpha)$$

$$B_a\theta = da \wedge \left(\theta_n \big|_{\partial M} \right) = 0$$

$$\Leftrightarrow \ \theta_n \big|_{\partial M} = 0$$

$$\Leftrightarrow \ B_r(*\theta) = (-1)^{n-1} *' \left(\theta_n \big|_{\partial M} \right) = 0$$

Green's Formula

$$\left(d\theta,\eta\right)=\left(\theta,\delta\eta\right)+\int_{\partial M}\left[B_r(\theta)\wedge B_r\left(*\eta\right)\right]_{n-1}$$

$$\delta_a{}^* = d$$

$$d^* = \delta_a$$

Graded Dirac Operators

- $D_+ = (d + \delta_a)_+ : \Omega_a^{\text{even}}(M) \to \Omega_a^{\text{odd}}(M)$
- $D_- = (d + \delta_a)_- : \Omega_a^{\text{odd}}(M) \to \Omega_a^{\text{even}}(M)$

(Euler characteristic operators)

The fundamental property of Dirac Operator

$$D^2 = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} D_- D_+ & 0 \\ 0 & D_+ D_- \end{pmatrix} = \begin{pmatrix} \Delta_a & 0 \\ 0 & \Delta_a \end{pmatrix}$$

$$\Delta_a = \delta_a d + d \delta_a$$

Index of a Graded Dirac Operator

$$\operatorname{ind} D = \dim \operatorname{Ker} D_{+} - \dim \operatorname{Ker} D_{-}$$

$$D = \begin{pmatrix} 0 & D_{-} \\ D_{+} & 0 \end{pmatrix} : \begin{matrix} \Omega_a^{\text{even}}(M) \\ \oplus \\ \Omega_a^{\text{odd}}(M) \end{matrix} \rightarrow \begin{matrix} \Omega_a^{\text{even}}(M) \\ \oplus \\ \Omega_a^{\text{odd}}(M) \end{matrix}$$

Index formula for the Euler characteristic operator

$$D_+ = (d+\delta_a)_+ : \Omega_a^{\text{even}}(M) \to \Omega_a^{\text{odd}}(M)$$

$$D_- = (d+\delta_a)_- : \Omega_a^{\text{odd}}(M) \to \Omega_a^{\text{even}}(M)$$

$\Rightarrow$

$$\begin{aligned}\operatorname{ind}(d+\delta_a) &= \dim \operatorname{Ker}(d+\delta_a)_+ - \dim \operatorname{Ker}(d+\delta_a)_- \\ &= \sum_{j=0}^{m} (-1)^j \dim H^j(M;\mathbf{R}) \\ &= \chi(M)\end{aligned}$$

Part IV

The relative de Rham Complex and Interior boundary value problems

Relative de Rham complex (1)

- X **compact manifold without boundary of dimension** n
- Y **compact submanifold of** X **of dimension** m
- $\iota : Y \to X$ **(natural inclusion map)**

Relative de Rham complex (2)

- $\imath : Y \to X$ **(natural inclusion map)**
- $\imath^* : \Omega^k(X) \to \Omega^k(Y)$ **(pull-back of $\imath$)**
- $\Omega^k(X, Y) = \left\{\theta \in \Omega^k(X) : \imath^*\theta = 0\right\}$

Relative de Rham complex (3)

$$\begin{array}{ccccccccc}
0 & \to & \Omega^{k-1}(X,Y) & \to & \Omega^{k-1}(X) & \to & \Omega^{k-1}(Y) & \to & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d' & & \\
0 & \to & \Omega^{k}(X,Y) & \to & \Omega^{k}(X) & \to & \Omega^{k}(Y) & \to & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d' & & \\
0 & \to & \Omega^{k+1}(X,Y) & \to & \Omega^{k+1}(X) & \to & \Omega^{k+1}(Y) & \to & 0
\end{array}$$

$$\iota^* d = d' \iota^*$$

Relative de Rham complex (4)

$$\Omega^{k-1}(X,Y) \xrightarrow{d_r^{k-1}} \Omega^{k}(X,Y) \xrightarrow{d_r^{k}} \Omega^{k+1}(X,Y)$$

$$d_r^{k}\, d_r^{k-1} = 0$$

Relative de Rham complex (5)

- $\left\{\alpha \in \Omega^k(X,Y) : d\alpha = 0\right\} = \operatorname{Ker} d^k$ **(closed forms)**
- $\left\{d\beta \in \Omega^k(X,Y) : \beta \in \Omega^{k-1}(X,Y)\right\} = \operatorname{Im} d^{k-1}$ **(exact forms)**
- $H^k(X,Y) = \operatorname{Ker} d^k / \operatorname{Im} d^{k-1}$

(de Rham cohomology group of X **relative to** Y**)**

Long exact sequence in de Rham cohomology

$$\rightarrow \quad H^{k-1}(Y) \quad \rightarrow \quad H^k(X,Y) \rightarrow H^k(X) \quad \rightarrow$$
$$\rightarrow \quad H^k(Y) \quad \rightarrow \quad H^{k+1}(X,Y) \rightarrow H^{k+1}(X) \rightarrow$$

Long exact sequence in simplicial cohomology

$$\to H^{k-1}(Y;\mathbf{R}) \to H^k(X,Y;\mathbf{R}) \to H^k(X;\mathbf{R}) \to$$

$$\to H^k(Y;\mathbf{R}) \to H^{k+1}(X,Y;\mathbf{R}) \to H^{k+1}(X;\mathbf{R}) \to$$

$$H^k(X,Y) \cong H^k(X,Y;\mathbf{R})$$

Analytic Approach
to
Relative de Rham Cohomology Theory

Sobolev Spaces

$$W_a(\mathbf{R}^n) = \left\{ u \in S'(\mathbf{R}^n) : \int_{\mathbf{R}^n} \left(1 + |\xi|^2\right)^a \left|\hat{u}(\xi)\right|^2 d\xi < \infty \right\}$$

$$\|u\|_a = \left(\int_{\mathbf{R}^n} \left(1 + |\xi|^2\right)^a \left|\hat{u}(\xi)\right|^2 d\xi \right)^{1/2}$$

Sobolev Spaces of Currents

$$W_a^p(X) = \left\{ u \in S'(X) : \alpha = \sum_{|I|=p} \alpha_I \, dx^I, \quad \alpha_I \in W_a(\mathbf{R}^n) \right\}$$

$$(\alpha, \beta)_{W_a^p(X)} = \int_X (I + \Delta)^{a/2} \alpha \wedge * \left((I + \Delta)^{a/2} \beta \right)$$

$*$: Hodge star operator on X

Δ : Laplace-Beltrami operator on X

$(I + \Delta)^{a/2}$: Fractional Power of $I + \Delta$

Operator D of Matrix Form

$$D = \begin{pmatrix} (d+\delta)_e & -(I+\Delta)^{-a}\left(\bullet \otimes \delta_Y\right) \\ i^* & 0 \end{pmatrix} : \begin{matrix} W_a^{\text{even}}(X) \\ \oplus \\ W_{-1}^{\text{odd}}(Y) \end{matrix} \to \begin{matrix} W_a^{\text{odd}}(X) \\ \oplus \\ W_1^{\text{even}}(Y) \end{matrix}$$

$$a = \frac{n-m}{2} = \frac{\operatorname{codim} Y}{2}$$

Domain of the Operator *D*

$$W_a^{\text{even}}(X) = \oplus W_a^{2j}(X)$$

$$W_a^{\text{odd}}(X) = \oplus W_a^{2j-1}(X)$$

$$W_1^{\text{even}}(Y) = \oplus W_1^{2i}(Y)$$

$$W_{-1}^{\text{odd}}(Y) = \oplus W_{-1}^{2i-1}(Y)$$

Interior Boundary Value Problems

$$\begin{pmatrix} \alpha \\ S \end{pmatrix} \in \mathrm{Ker}\ \mathrm{D}$$

$$\Leftrightarrow$$

$$\begin{cases} (I + \Delta)^{a}(d + \delta)\alpha = S \otimes \delta_{Y} & \text{on } X \\ i^{*}\alpha = 0 & \text{on } Y \end{cases}$$

Long exact sequence in relative de Rham cohomology

$$\to \quad H^{2p-1}(Y) \quad \to \quad \mathrm{Ker}^{2\mathrm{p}}\ D \to H^{2p}(X) \quad \to$$

$$\to \quad H^{2p}(Y) \quad \to \quad \mathrm{Ker}^{2p+1}\ D^{*} \quad \to \quad H^{2p+1}(X) \to$$

Five lemma

$$\mathrm{Ker}^{2\mathrm{p}}\ D \cong H^{2p}(X,Y)$$

$$\mathrm{Ker}^{2\mathrm{p}+1}\ D^{*} \cong H^{2p+1}(X,Y)$$

Index formula for relative de Rham complex

$$\operatorname{ind} D = \dim \operatorname{Ker} D - \dim \operatorname{Ker} D^*$$

$$= \sum_{j=0}^{m} (-1)^j \dim H^j(X, Y; \mathbf{R})$$

$$= \chi(X, Y) = \chi(X) - \chi(Y)$$

Concrete Examples

Example 1 (1)

$$\dim X = 4, \quad \dim Y = 2$$

$$\begin{cases} W_1^{even}(X) = W_1^0(X) \oplus W_1^2(X) \oplus W_1^4(X) \\ W_1^{odd}(X) = W_1^1(X) \oplus W_1^3(X) \end{cases}$$

$$\begin{cases} W_1^{even}(Y) = W_1^0(Y) \oplus W_1^2(Y) \\ W_{-1}^{odd}(Y) = W_{-1}^1(Y) \end{cases}$$

Example 1 (2)

$$D = \begin{pmatrix} (d+\delta)_e & -(I+\Delta)^{-1}\left(\bullet\otimes\delta_Y\right) \\ i^* & 0 \end{pmatrix} : \begin{matrix} W_1^{\text{even}}(X) \\ \oplus \\ W_{-1}^{\text{odd}}(Y) \end{matrix} \to \begin{matrix} W_1^{\text{odd}}(X) \\ \oplus \\ W_1^{\text{even}}(Y) \end{matrix}$$

Example 1 (3)

$$D\begin{pmatrix}\alpha_0\\ \alpha_2\\ \alpha_4\\ S\end{pmatrix}=\begin{pmatrix}d\alpha_0+\delta\alpha_2-(I+\Delta)^{-1}(S\otimes\delta_Y)\\ d\alpha_2+\delta\alpha_4\\ i^*\alpha_0\\ i^*\alpha_2\end{pmatrix}$$

Example 1 (4)

$$\operatorname{ind} D = \chi(X) - \chi(Y)$$

$$= \frac{1}{32\pi^2}\int_X \left(\kappa^2 - 4\|\mathrm{Ric}\|^2 + \|R\|^2\right)\mu_X - \frac{1}{2\pi}\int_Y K\,\mu_Y$$

Curvatures

- $R = R_{ikj\ell}$ **Riemannian curvature tensor**
- $\mathrm{Ric} = R_{ikjk}$ **Ricci curvature**
- $\kappa = R_{ijij}$ **Scalar curvature**
- K **Gaussian curvature**

Example 2 (1)

$$\dim X = 6, \quad \dim Y = 4$$

$$\begin{cases} W_1^{even}(X) = W_1^0(X) \oplus W_1^2(X) \oplus W_1^4(X) \oplus W_1^6(X) \\ W_1^{odd}(X) = W_1^1(X) \oplus W_1^3(X) \oplus W_1^5(X) \end{cases}$$

$$\begin{cases} W_1^{even}(Y) = W_1^0(Y) \oplus W_1^2(Y) \oplus W_1^4(Y) \\ W_{-1}^{odd}(Y) = W_{-1}^1(Y) \oplus W_{-1}^3(Y) \end{cases}$$

Example 2 (2)

$$D = \begin{pmatrix} (d+\delta)_e & -(I+\Delta)^{-1}\left(\bullet \otimes \delta_Y\right) \\ i^* & 0 \end{pmatrix} : \begin{matrix} W_1^{\text{even}}(X) \\ \oplus \\ W_{-1}^{\text{odd}}(Y) \end{matrix} \to \begin{matrix} W_1^{\text{odd}}(X) \\ \oplus \\ W_1^{\text{even}}(Y) \end{matrix}$$

Example 2 (3)

$$\operatorname{ind} D = \chi(X) - \chi(Y)$$

$$= \frac{1}{384\pi^3}\int_X (\kappa^3 - 12\kappa\|\mathrm{Ric}\|^2 + 3\kappa\|R\|^2 + 16R^i_j R^j_k R^k_i$$

$$+24R^{ij}R^{k\ell}R_{ikj\ell} - 24R^{ij}R_{iabc}R_j^{abc} - 8R^{ijk\ell}R^{rs}_{ik}R_{js\ell r} + 2R^{ijk\ell}R_{ijrs}R^{rs}_{k\ell})\mu_X$$

$$-\frac{1}{32\pi^2}\int_Y \left(\kappa_Y^2 - 4\|\mathrm{Ric}_Y\|^2 + \|R_Y\|^2\right)\mu_Y$$

Index formula
of
Agranovic-Dynin type

Vishik Operator P

$$P\varphi := i^{*}\left(G(I+\Delta)^{-a}\left(\varphi \otimes \delta_{Y}\right)\right), \quad \varphi \in \Omega^{p}(Y)$$

$$G := \frac{1}{2\pi i}\int \frac{1}{z}\,(zI-\Delta)^{-1}\,dz$$ **Green operator**

$P \in L_{\mathrm{cl}}^{-2}(Y)$ **classical, elliptic pseudo - differential operator of order** -2

Index formula of Agranovic-Dynin type

$$\operatorname{ind}\left(d+\delta\right)_e - \operatorname{ind}\begin{pmatrix} (d+\delta)_e & -(I+\Delta)^{-a}\left(\bullet\otimes\delta_Y\right) \\ i^* & 0 \end{pmatrix}$$

$$= \operatorname{ind}\left(d' + P\delta' P^{-1}\right)_e$$

END