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Homotopy theory of higher categories. From Segal categories to n-categories and beyond. (English) Zbl 1232.18001

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The theory of n-categories is currently under active research by a batch of different schools around the world, though the crucial definition of weakly associative higher category still remains elusive or sometimes even illusory. Its history goes back to separate but interrelated tracks in algebraic topology, algebraic geometry and category theory. This book on the theory of higher categories is firmly based upon the notion of Segal category, which is a kind of category enriched over simplicial sets. It was *Z. Tamsamani* [K-Theory 16, No. 1, 51–99 (1999; Zbl 0934.18008)] who applied this method to n-categories for the first time. The guiding principle is to use the category $\triangle$ of simplicies as the basis for all the higher coherency conditions coming in when allowing weak associativity. A Segal $(n+1)$-category is a functor from $\triangle^0$ to the category of Segal n-categories, whose zeroth element is a discrete set, and such that the Segal maps are equivalences. This iterative point of view towards higher categories is the main topic of the present book. The notion of Segal category undoubtedly goes back to *G. Segal* [Topology 13, 293–312 (1974; Zbl 0284.55016)], but its first modern treatments are *W. G. Dwyer*, *D. M. Kan* and *J. H. Smith* [J. Pure Appl. Algebra 57, No. 1, 5–24 (1989; Zbl 0678.55007)] and *R. Schwänzl* and *R. Vogt* [Bol. Soc. Mat. Mex., II. Ser. 37, No. 1–2, 431–448 (1992; Zbl 0853.55010)].

Just as one calls two homotopic maps between two spaces the same in homotopy theory, it is often necessary to consider two naturally equivalent functors the same in category theory. Starting in the 1950s and 1960s, the notion of derived category, an abelianized version of the homotopical construction, became crucial to a number of areas in homological algebra as well as in algebraic geometry. In the abelian case, the notion of localization of a category was proposed by Serre, as can be seen in [*A. Grothendieck*, Tohoku Math. J., II. Ser. 9, 119–221 (1957; Zbl 0118.26104)]. Its systematic treatment is the subject of Gabriel and Zisman's book [*P. Gabriel* and *M. Zisman*, Calculus of fractions and homotopy theory. Berlin-Heidelberg-New York: Springer-Verlag (1967; Zbl 0186.56802)]. In the nonabelian case [*D. G. Quillen*, Homotopical algebra. Lecture Notes in Mathematics. 43. Berlin-Heidelberg-New York: Springer-Verlag (1967; Zbl 0168.20903)], the notion of model category adequately formulates a good collection of requirements that can be made on the intermediate categorical data. In the far future one could hopefully start directly with a notion of higher category and bypass the model-categoric step entirely, which would result in intractable questions of bootstrapping at present.

Part I, consisting of 6 chapters, begins with a review on history and motivation concerning higher categories (Chapter 1). Chapter 2 is devoted to strict n-categories, which are shown to be insufficient by any means. Chapter 3 studies a certain number of basic elements expected of any theory of n-categories, and which can be explained without referring to a full definition. Chapter 4 is a hasty review of one of the main directions towards the theory of higher categories consisting of a number of operadic approaches, which can be seen, e.g., in [*J. C. Baez* and *J. Dolan*, Adv. Math. 135, No. 2, 145–206 (1998; Zbl 0909.18006)] and [*M. A. Batanin*, Adv. Math. 136, No. 1, 39–103 (1998; Zbl 0912.18006)]. Chapter 5 is a quick review of simplicial approaches to higher categories, including $(\infty, 1)$-categories. Returning to the main stream of the book, Chapter 6 tells how one can abstract Tamsanami's [Zbl 0934.18008] iteration process to obtain a theory of $\mathcal{M}$-enriched categories, weak in Segal's sense, for model category $\mathcal{M}$.

Part II, consisting of three chapters, deals with categorical preliminaries. It begins with Chapter 7, which reviews some of the basic elements of Quillen's theory of model categories [Zbl 0168.20903] and their modern variants. Chapter 8 begins with a review of the theory of locally presentable categories based on the book [*J. Adámek* and *J. Rosický*, Locally presentable and accessible categories. London Mathematical Society Lecture Note Series. 189. Cambridge: Cambridge University Press (1994; Zbl 0795.18007)]. The applicability of the theory to model categories comes from J. Smith's notion of combinatorial model category (see *T. Beke* [Math. Proc. Camb. Philos. Soc. 129, No. 3, 447–475 (2000; Zbl 0964.55018), J. Pure Appl. Algebra 164, No. 3, 307–324 (2001; Zbl 1002.18012)], *D. Dugger* [Adv. Math. 164, No. 1, 177–201 (2001; Zbl 1001.18001)] and *J. Rosický* [Appl. Categ. Struct. 17, No. 3, 303–316 (2009; Zbl 1175.55013)]), slightly modified by *C. Barwick* [Homology Homotopy Appl. 12, No. 2, 245–320 (2010; Zbl 1243.18025)]. The modern point of view allows one to see any transfinite composition of pushouts as being a cell complex in a generalized sense. *P. S. Hirschhorn* [Model categories and their localizations. Mathematical Surveys and Monographs. 99. Providence, RI: American Mathematical Society (AMS) (2003; Zbl 1017.55001)] formalized this idea for the small object argument and left Bousfield localization in the context of a general locally presentable categories, though using an additional assumption of a monomorphism property of the generating set of arrows $I \subset ARR(\mathcal{M})$, encoded in his notion of cellular model category. The author avoids this hypothesis by using a somewhat more abstract approach to cell complexes. The author's discussion covers much the same material as [*J. Lurie*, Higher topos theory. Annals of Mathematics Studies 170. Princeton, NJ: Princeton University Press (2009; Zbl 1175.18001)] appendix, which introduced the notion of a tree generalizing the standard transfinite cell-addition process. The basic idea is that once we have attached a certain number of cells, the next cell is attached along a κ-presentable subcomplex. The author sticks to the standard ordinal presentation, but introduces a category of inclusions of cell complexes and shows that the category of κ-small inclusions of cell complexes into a given one is κ-filtered. A proof of Lurie's theorem (A.1.5.12 of [Zbl 1175.18001]) that cofibrations are cell complexes over κ-small cofibrations rather than just retracts of such is sketched. The author turns to a main application of this result to construct a generating set for injective cofibrations. The chapter is concluded by the construction of a cofibrantly generated model structure, which enables one in later chapters to apply it without having to come back to the cardinality argument. The reviewer calls this chapter most technical in the book. Chapter 9 considers a special case of left Bousfield localization in which everything is much more explicit.

Part III, consisting of 6 chapters, is concerned with the calculus of generators and relations. The first chapter (Chapter 10) deals with precategories, which are a kind of simplicial objects without imposing the Segal conditions. The passage from a precategory to a weakly enriched category consists of enforcing the Segal conditions using the small object argument. The main work of Chapter 11 is to try to enforce the cartesian product condition on an arbitrary functor $\Phi \to \mathcal{M}$ for the case of homotopy realizations in a model category $\mathcal{M}$. Using the model category for algebraic theories developed in Chapter 11, Chapter 12 gets model structures for Segal precategories on a fixed set of objects and turns to the global category $\mathcal{PC}(\mathcal{M})$ of $\mathcal{M}$-precategories with arbitrary variable set of objects. The study of $\mathcal{PC}(\mathcal{M})$ is continued in Chapter 13, where various classes of cofibrations are defined and discussed. In Chapter 14 the author looks more closely at the specific calculus of generators and relations corresponding to the direct left Bousfield localization of $\mathcal{PC}(X, \mathcal{M})$ considered in Chapter 12. Chapter 15 considers one of the main examples of the theory, say, when $\mathcal{M} = \mathcal{K}$ is the Kan-Quillen model category of simplicial sets. The notion of weakly $\mathcal{M}$-enriched category then becomes the notion of Segal category, one version of the $(\infty, 1)$-categories which occupies a ubiquitous position in applications of higher category theory.

To prove that the classes of Reedy cofibrations and global weak equivalences give a model structure on $\mathcal{PC}(\mathcal{M})$, some computational work is indispensible. Part IV, consisting of three chapters, is concerned with this computation. The first chapter (Chapter 16) studies the categories with an ordered set of objects $x_0, \dots, x_n$ and morphisms other than the identity from x_i to x_j only when $i < j$. The main computation, which is what happens when one takes the product of two such categories, is presented in Chapter 17. Chapter 18 establishes the contractibility of $\Xi(N \mid N')$ in two steps:first considering the case of Segal categories; then going to the case of $\mathcal{M}$-enriched weak categories by transfer along $\mathcal{K} \to \mathcal{M}$. Chapter 19 concludes the proof without no further obstacles.

Theorem 19.3.2, claiming that, given a tractable left proper cartesian model category $\mathcal{M}$, the model category $\mathcal{PC}_{\text{Reedy}}(\mathcal{M})$ of $\mathcal{M}$-enriched precategories with Reedy cofibrations and global weak equivalences is so, paves the way to Part V, which consists of four chapters. The first chapter (Chapter 20) sets up the basic context and notation, and relates higher groupoids back to homotopy theory. Chapter 21 generalizes the large body of knowledge on usual category theory to the higher categorical context. Chapter 22 considers the particular question of constructing limits and colimits of higher categories themselves. Chapter 23 concludes this part as well as the book by looking at the Baez-Dolan stabilization hypothesis. *J. C. Baez* and *J. Dolan* [J. Math. Phys. 36, No. 11, 6073–6105 (1995; Zbl 0863.18004)] proposed a whole series of definitions and properties to be expected of a good theory of n-categories. The main points were their conjectures on the universal property of certain n-categories defined by looking at cobordisms, which generalize the well-known ideas of topological and conformal field theories associated to knot invariants. The main result of the chapter is that if $\mathcal{A}$ is a k-uply monoidal n-category and if $k \geq n + 2$, then there exists a delooping $\mathcal{Y}$ of $\mathcal{A}$ which is a $(k+1)$-uply monoidal n-category. A related result can be seen in *J. Lurie*'s expository paper [Somerville, MA: International Press. 129–280 (2009; Zbl 1180.81122)], which is highly readable and greatly suggestive.

All in all, the book is well written and highly readable. Having a good theory of n-categories paves a way to pursuing any of the several programs such as Grothendieck's famous "Pursuing stacks," the generalization to n-stacks and n-gerbes of the work of *L. Breen* [On the classification of 2-gerbes and 2-stacks. Astérisque. 225. Paris: Société mathématique de France (1994; Zbl 0818.18005)], or the program of Baez and Dolan in topological quantum field theory [loc. cit., Zbl 0863.18004].

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MSC:

18-02 Research monographs (category theory)
18D05 Double categories, 2-categories, bicategories and generalizations
18G55 Nonabelian homotopical algebra
55P99 Homotopy theory
55U40 Topological categories, foundations of homotopy theory
55U35 Abstract homotopy theory; axiomatic homotopy theory

Keywords:

higher category theory; Segal's delooping machine; quasicategory; homotopy theory; operad; Baez-Dolan stabilization hypothesis; localization; cobordism; topological field theory; conformal field theory; knot invariant; groupoid; model category; derived category; Reedy cofibration; simplicial category; simplicial set; Segal map; tractable model category; Smith's recognition theorem; nonabelian cohomology; stack; gerb; braided monoidal category; Segal precategory; Segal category; left Bousfield localization; combinatorial model category; cell complex; locally presentable category