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Monoidal functors, species and Hopf algebras. (English) Zbl 1209.18002
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In 2006 the authors published a wonderful monograph on combinatorial Hopf algebras [*M. Aguiar* and *S. Mahajan*, Coxeter groups and Hopf algebras. Fields Institute Monographs 23 (2006; Zbl 1106.16039)], in which they used geometric methods, inspired by Jacques Tits' projection operators in the theory of buildings, to study various Hopf algebras arising in combinatorics. Now they make a great step towards a deeper understanding of combinatorial Hopf algebras in this voluminous monograph consisting of three parts.

Part I, consisting of seven chapters, is concerned with monoidal categories. Chapter 1 reviews some basic notions related to monoidal categories, such as braided monoidal categories, bimonoids and Hopf monoids, Lie monoids, unbracketed and unordered tensor products, and the internal Hom functor. Chapter 2 is devoted to graded vector spaces as an illustration of the theory described in Chapter 1. In Chapter 3 the authors investigate in great detail the notion of a bilax monoidal functor on a braided monoidal category, namely, a functor which is both lax monoidal and colax monoidal abiding by appropriate compatibility conditions with respect to the two structures. The gist of their arguments is that a bilax monoidal functor preserves bimonoids, just as a lax monoidal functor preserves monoids while a colax monoidal functor preserves comonoids. Most bilax monoidal functors considered in this monograph preserve Hopf monoids as well. The crux of this chapter is the analogies between the notion of associative monoid and that of lax monoidal functor, and between the notion of commutative monoid and that of braided lax monoidal functor. Chapter 4 extends these analogies to other types of monoids besides "associative" and "commutative" ones, treating these notions in full generality in terms of operads. Chapter 5 discusses the most classical example of a bilax monoidal functor in the construction of a chain complex from a simplicial module, showing that the classical maps of Eilenberg-Zilber and Alexander-Whitney provide the lax and colax structures which turn it into a bilax monoidal functor. Chapter 6, viewing bilax monoidal functors as functors not between braided monoidal categories but rather between 2-monoidal categories, considers the Eckmann-Hilton argument from a categorical viewpoint, in which a strong 2-monoidal category is equivalent to a braided monoidal category [*A. Joyal* and *R. Street*, Adv. Math. 102, No. 1, 20–78 (1993; Zbl 0817.18007)], a double monoid is equivalent to a commutative monoid, and a double lax monoidal functor is equivalent to a braided lax monoidal functor. In Chapter 7 the authors climb up the ladder of higher monoidal categories, showing that there are $n + 1$ different types of functors between two n-monoidal categories while there are $n + 1$ different types of monoids in an n-monoidal category.

Part II, consisting of 7 chapters, is the heart of this monograph, in which *A. Joyal*'s species [Adv. Math. 42, 1–82 (1981; Zbl 0491.05007)] play a prominent role. It is devoted to a careful study of the monoidal category of species, Hopf monoids therein, and the discussion of several examples. The ultimate goal is to provide a solid conceptual framework for the study of a large number of Hopf algebras of a combinatorial nature, including the Hopf algebra of symmetric functions, quasi-symmetric functions, noncommutative symmetric functions, and so on. The central dogma in the authors' approach is that a proper understanding of these objects and their interrelationships requires the consideration of a more general setting, namely, that of Hopf monoids in the monoidal category of species. In Chapter 8 the authors enter the world of species, which are graded vector spaces whose degree n component is equipped with an action of the symmetry group S_n for each n. Chapter 9 discusses a one-parameter deformation of the braiding on the category of species equipped with the Cauchy product. Chapter 10 ties Coxeter theory to species, so that the authors confine their considerations to the symmetry group, the Coxeter group of type A. The principal objective in Chapter 11 is to understand the relations between various universal objects. The goal of Chapter 12 is to construct Hopf monoids in species motivated by the geometry of the Coxeter group of type A. In Chapter 13 the authors discuss a number of examples of Hopf monoids arising naturally in combinatorics. Many of these ideas go back to the paper of *S. A. Joni* and *G.-C. Rota* [Contemp. Math. 6, 1–47 (1982; Zbl 0491.05021)]. Chapter 14 is devoted to colored species, which are a higher-dimensional generalization of species. Roughly speaking, colored species correspond to multigraded vector spaces in the same way as species correspond to graded vector spaces.

Part III, consisting of six chapters, is concerned with the relavant bilax functors on species, called Fock functors. They are used to obtain bialgebras, which are then shown to be the desired Hopf algebras. Chapter 15 formulates the constructions in *C. R. Stover* [J. Pure Appl. Algebra 86, No. 3, 289–326 (1993; Zbl 0793.16016)]; *F. Patras* and *C. Reutenauer*, Mosc. Math. J. 4, No. 1, 199–216 (2004; Zbl 1103.16026)]; *F. Patras* and *M. Schocker* [Adv. Math. 199, No. 1, 151–184 (2006; Zbl 1154.16029); J. Algebr. Comb. 28, No. 1, 3–23 (2008; Zbl 1180.05032)], and studies their properties in categorical terms. Hopf monoids in the category of species are richer than their corresponding graded Hopf algebras. In Chapter 16 the authors, generalizing the theory in Chapter 15, show that the q-Fock functors can be applied to p-deformations of Hopf monoids in which case one obtains pq-deformations of the corresponding Hopf algebras. Chapter 17 applies the Fock functors to the various Hopf monoids constructed in Chapters 12 and 13 so as to obtain the corresponding Hopf algebras, many of which have appeared in the literature [cf. *M. Hazewinkel*, Acta Appl. Math. 75, No. 1–3, 55–83 (2003; Zbl 1020.16027) and Acta Appl. Math. 85, No. 1–3, 319–340 (2005; Zbl 1139.16026)]. In Chapter 18 the authors describe the adjoints of the Fock functors whenever they exist, enriching our understanding of the interplay between species and graded vector spaces. In Chapter 19 the various Fock functors and their deformations, which are bilax functors from species to graded vector spaces, are generalized, depending on a vector space (the space of decorations). The earlier theory generalizes in a straightforward way to this more general setting. The principal goal in Chapter 20 is the construction of multivariate versions of the Fock functors in Chapters 15 and 16, which is accomplished by replacing species by colored species and graded vector spaces by multigraded vector spaces.

The monograph, consisting of 784 pages, is accompanied by four appendices on categorical preliminaries, operads, the looping principle and the simplicial category. It is prefaced by the forward of Kenneth Brown and Stephen Chase (the authors' former thesis advisors) and the one of André Joyal. This monograph is indeed a monumental work at the crossroads between category theory, algebra and combinatorics.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18-02	Research monographs (category theory)
18D10	Monoidal, symmetric monoidal and braided categories
18D05	Double categories, 2-categories, bicategories and generalizations
18D20	Enriched categories (over closed or monoidal categories)
18D25	Strong functors, strong adjunctions
18D50	Operads
16T05	Hopf algebras and their applications
16-02	Research monographs (associative rings and algebras)
05A10	Combinatorial functions
05E99	Algebraic combinatorics

Keywords:

Fock functor; Hopf monoid; species; Hopf algebra; combinatorics; colored species; graded vector space; bilax monoidal functor; braided monoidal category; bimonoid; Cauchy product; Eckmann-Hilton argument; Hadamard product; Jacobi identity; simplicial category; Stover's construction; Eilenberg-Zilber map; operad; higher monoidal category; pseudomonoid; Alexander-Whitney map; Coxeter theory; multigraded vector space; looping principle