Bautista, R.; **Salmerón, L.**; **Zuazua, R.**

Differential tensor algebras and their module categories. (English) Zbl 1266.16007

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An elegant approach to matrix problems through differential graded categories was proposed by *A. V. Roiter* and *M. M. Kleiner* [Zbl 0356.16011], where the natural correspondence between normal bocses (bocs is an acronym for "bimodule over a category with coalgebra structure") and differential graded categories was established. The basic idea there was to consider an interesting class of problems admitting a generalization of the representations of quivers introduced by *P. Gabriel* [in Zbl 0232.08001]. The underlying strategy was to make use of representations of a new type of combinatorial nature called bigraphs. The subject is studied not only in the Kiev school in representation theory but also in the English school (see, e.g., [Zbl 0777.16009]; [Zbl 0734.16004] and [Zbl 0661.16026]) and the Mexican school (see, e.g., [Zbl 1113.16021]).

This monograph is engaged in ditalgebras (ditalgebra is an acronym for "differential tensor algebra") and their categories of modules, including reduction techniques, which have proved to be highly useful in the theory of representations of finite-dimensional algebras. The authors prefer working in the concrete context of ditalgebras to doing in more abstract category-theoretic guises, though some applications such as in the study of coverings make category-theoretic considerations inevitable. The authors also prefer making use of some special dual basis and reduction by an admissible module to exploiting bocses with underlying additive categories and pullback reduction constructions. The reader is assumed to be familiar with rudiments of the theory of representations of algebras and homological algebra (e.g., [Zbl 0863.18001] and [Zbl 1093.20003] will suffice).

The monograph consists of 36 sections. We look closely at some of them.

Although the category of $\mathcal{A}$-modules is not necessarily Abelian for a ditalgebra $\mathcal{A}$, the triangularity of $\mathcal{A}$ enables one to use elementary inductions in the study of $\mathcal{A}-\mathrm{Mod}$, which is the main topic of §5. In particular, the significant notion of a Roiter ditalgebra is introduced here. The contents of §6, where a natural exact structure for the category of modules over a Roiter ditalgebra is presented, are based largely upon [Zbl 1067.16011]. The existence of almost split conflations for bocses under some finiteness conditions, which was established independently in [Zbl 0777.16009] and [Zbl 0743.16013] by making use of the main result of [Zbl 0457.16017], is shown in §7, where the authors dispense with the Auslander-Smalø theorem. In §9 the authors explore necessary conditions under which a functor $F\colon \mathcal{A}'-\mathrm{Mod} \to \mathcal{A}-\mathrm{Mod}$ between module categories of layered ditalgebras reflects the property of being Roiter. After discussing hom-tensor relations and dual basis in §11, the important notion of an admissible module is introduced in §12, where, given an admissible module X over a proper subditalgebra of a layered ditalgebra $\mathcal{A}$, the authors construct a new layered ditalgebra $\mathcal{A}^X$ and its associated functor $F^X\colon \mathcal{A}^X-\mathrm{Mod} \to \mathcal{A}-\mathrm{Mod}$. Their basic properties are discussed in the succeeding five sections (§13-§17). In §13 the authors examine sufficient conditions for F^X to be full and faithful. The main result in §14 is that, given an initial subditalgebra $\mathcal{A}'$ of a triangular ditalgebra $\mathcal{A}$, the associated ditalgebra $\mathcal{A}^X$ is triangular, as far as we assume some triangularity condition on the admissible $\mathcal{A}'$-module X. Introducing the notion of a free triangular ditalgebra $\mathcal{A}$, §15 gives sufficient conditions on an admissible module X yielding free triangular ditalgebra $\mathcal{A}^X$. The main result in §16 is that, given a complete triangular admissible $\mathcal{A}'$-module X with an initial subditalgebra $\mathcal{A}'$ of the Roiter ditalgebra $\mathcal{A}$, we get a Roiter ditalgebra $\mathcal{A}^X$. §17 is concerned with general settings in which admissible modules occur. The contents of §18 are refreshingly borrowed from [Zbl 1081.16025]. In §19 the authors construct, for each finite-dimensional algebra Λ admitting a splitting pair (S, J) with $J = \operatorname{rad}\Lambda$, a finite-dimensional Roiter ditalgebra $\mathcal{D}^\Lambda$ and an equivalence of categories $\Xi_\Lambda\colon \mathcal{D}^\Lambda-\mathrm{Mod} \to \mathcal{P}^1(\Lambda)$, which are an adaption of those in [Zbl 0661.16026] to the present context of ditalgebras.

In the representation theory of finite-dimensional algebras, the notions of finiteness, tameness and wildness play a significant role. This monograph provides a fresh point of view of well-known results on tame and wild ditalgebras, on tame and wild algebras, and on their modules, together with some new results and some new proofs. Introducing the notion of wildness, §22 provides some examples and some useful criteria for wildness. Introducing the notions of nested and seminested ditalgebras and their associated marked bigraphs, §23 investigates important operations on seminested ditalgebras such as reduction of an edge, deletion of idempotents, regularization and unravelling of a loop, etc. We note that nested ditalgebras stand close to almost free bocses of *Yu. A. Drozd* [Zbl 0454.16014], while seminested ditalgebras lie close to layered bocses in [Zbl 0661.16026] and [Zbl 0741.16005]. Defining critical ditalgebras, §24 establishes that they are wild. The main result in §26 is that the study of $\mathcal{A}$-E-bimodules with bounded length of a non-wild seminested ditalgebra $\mathcal{A}$ over an algebraiclly closed field is largely reducible to the study of bimodules over a finite number of minimal ditalgebras. Drozd's tame and wild theorem, established in [Zbl 0454.16014], is presented in §27 on the lines of [Zbl 0304.08006]. The contents of §28 are largely borrowed from [Zbl 1170.16008]. The main purpose of §29 is to demonstrate that the study of Λ-modules of bounded length over a non-wild finite-dimensional algebra Λ with coefficients in an algebraically closed field, is largely reducible to the study of modules over one minimal ditalgebra. Introducing the notion of absolute wildness, §30 shows that, as far as the coefficient fields are algebraically closed, the introduced notion is really equivalent to that of wildness. The main result in §31 is that the notion of tameness and that of generic tameness are equivalent, as far as finite-dimensional algebras over algebraically closed fields are concerned. The main result in §32 is Crawley-Boevey's structure theorem for Auslander-Reiten sequences of tame finite-dimensional algebras over algebraically closed fields. The presentation in §33 is developed partly from [Zbl 1009.16015]. The authors devote §34 to investigating an important family of nested ditalgebras, namely, the ditalgebras associated with posets, providing an explicit relationship between the category of representations of a poset $\mathbb{S}$ and the category of modules over its associated ditalgebra $\mathcal{A}^{\mathbb{S}}$ without any assumption on finite-dimensionality. Representations of posets were introduced in [Zbl 0336.16031], while representations of pairs of partially ordered sets were introduced in [Zbl 0345.06001]. Tame posets were classified in [Zbl 0362.06001]; [Zbl 0461.16024] and [Zbl 0452.16020], while tame pairs of posets were classified in [Zbl 0659.16021]. The formulation of the problem of pairs of posets in terms of differential graded categories is due to [Zbl 0356.16011], while the idea of realizing categories of representations of posets as subcategories of module categories originated in [Zbl 0707.16003]. The next section (§35) consists of lemmas exhibiting some interesting families of wild seminested ditalgebras with one or two points. The authors' proofs rest on the use of appropriate quotients and reductions exploiting admissible modules in place of producing explicit bimodules for wildness as in [Zbl 1104.16014]. The last section (§36) is generously engaged in providing answers to selected exercises.

Let alone the clumsiness of the authors' English, the monograph is well written, and will inspire the study on bocses and related structures.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

16E45	Differential graded algebras and applications	Cited in **2** Reviews
16D90	Module categories (associative rings and algebras); Morita equivalence and duality	Cited in **11** Documents
16-02	Research monographs (associative rings and algebras)	
18E30	Derived categories, triangulated categories	
16G20	Representations of quivers and partially ordered sets	
16G60	Representation type of noncommutative rings and modules	

Keywords:

differential tensor algebras; differential graded categories; categories of modules; bigraphs; reductions; admissible modules; representations of posets; tameness; wildness; Drozd tame and wild theorem; Crawley theorem; layers; bocses; finite-dimensional algebras; representation types