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Modern classical homotopy theory. (English) Zbl 1231.55001

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Classical homotopy theory emerged in the 1950s and was later largely codified in the abstract notion of a model category. This includes the notions of fibration and cofibration, CW complexes, long fiber and cofiber sequences, loop space, suspension, and so on. The basic idea of homotopy theory is that many topological problems as well as their solutions do not change if the maps involved are replaced by their continuous deformations. Homotopy theory is the study of the functor $\mathbf{Top} \to H\mathbf{Top}$ sending topological entities to their homotopical counterparts. The aim of this book is to develop classical homotopy theory and some important developments that flow from it, using the more modern techniques of homotopy limits and colimits, so that homotopy pushouts and homotopy pullbacks play a central role.

As *E. H. Brown*'s representability theorem ["Cohomology theories", Ann. Math. (2) 75, 467–484 (1962; Zbl 0101.40603)] shows, homology and cohomology are contained in classical homotopy theory. The author undoubtedly has written the book with the theory of model categories firmly in mind, but he has chosen to work with spaces throughout, happily making use of results that are indigenous to the realm of spaces. The author generally uses topological or homotopy-theoretic arguments rather than algebraic ones in order to let the balance between algebra and topology lean towards topology.

The remarkable feature of the book is that there are no outright proofs of theorems, which are generally followed by multi-part problems forcing the student to find proofs by himself or herself. In addition, there are numerous exercises, which are intended to help the student develop some habits of mind in reading mathematics. All in all, the book is a good textbook on homotopy theory, leading the student familiar only with rudiments of algebra and topology to its current frontiers.

This monograph consists of seven parts with an appendix on algebra. Part 1, consisting of two chapters, is concerned with the fundamentals of category theory.

Part 2, consisting of 8 chapters, deals with semi-formal homotopy theory. The main result of Chapter 3, wisely taken for granted, is that there is a nice category of topological spaces, which contains all CW complexes, which is closed under limits and colimits, and which is cartesian closed. Theorem 3.16 is by no means a theorem but a problem. The final chapter of Part 2 presents model categories axiomatized by Quillen, Kan and others, although the author has no intention to develop the entire abstract theory. The other chapters of Part 2 are concerned with such topics as cofibration and fibration, homotopy limits and colimits, homotopy pushouts and pullback squares, etc.

Part 3, consisting of 5 chapters, begins with Chapter 11, which, exploring the relation between n-equivalence and n-connectivity, establishes the celebrated Whitehead theorem [*J. H. C. Whitehead*, "Combinatorial homotopy. I", Bull. Am. Math. Soc. 55, 213–245 (1949; Zbl 0040.38704); "Combinatorial homotopy. II", ibid. 55, 453–496 (1949; Zbl 0040.38801)]. Chapter 12 investigates the uses and implications of the simple observation that an n-cell can be subdivided into smaller n-cells, applying the idea to establish the Seifert-van Kampen theorem and the cellular approximation theorem. The underlying theme of Chapter 13 is that a map which is locally a fibration (or a weak fibration) is really a fibration (or a weak fibration). The theorem was first established by *W. Hurewicz* ["On the concept of fiber space", Proc. Natl. Acad. Sci. USA 41, 956–961 (1955; Zbl 0067.15902)] and was given its definitive form by *A. Dold* ["Partitions of unity in the theory of fibrations", Proc. Int. Congr. Math., Stockholm 1962, 459–461 (1963; Zbl 0203.25501)]. Probing into the point-set topology of cofibrations, Chapter 14 establishes the interesting result that pulling back a cofibration along a fibration results in a cofibration. The final chapter of Part 3 is concerned with a variety of topics related to or based upon the preceding chapters such as the sections on locally trivial bundles.

Part 4, consisting of 5 chapters, begins with Chapter 16 establishing the existence of cellular replacements, which is used to detect connectivity. Here Postnikov sections are constructed and it is shown that any topological group is homotopically equivalent to a loop space. Chapter 17 is concerned with the construction by *I. M. James* ["Reduced product spaces", Ann. Math. (2) 62, 170–197 (1955; Zbl 0064.41505)], which leads to the Freudenthal suspension theorem [*H. Freudenthal*, "Über die Klassen der Sphärenabbildungen. I. Große Dimensionen", Compos. Math. 5, 299–314 (1937; Zbl 0018.17705; JFM 63.1161.02)] and a variety of computations. The existence and properties of Eilenberg-MacLane spaces are established. Chapter 18 resolves the two basic questions: how close is a homotopy pullback square to being a homotopy pushout square and how close is a homotopy pushout square to being a homotopy pullback square? Chapter 19 shows that maps from one wedge of n-spheres to another are fully described by certain integer matrices, which is used to define and study Moore spaces and to produce a noncontractible CW complex with trivial suspension. The smash product pairing of homotopy groups is defined and used to express the smallest nontrivial homotopy group of a smash product in terms of the homotopy groups of the factors. Chapter 20 deals with the long-promised example of a prepushout diagram which has no pushout in the homotopy category. Certain very tightly controlled infinite cone decompositions with respect to Moore spaces are investigated. The amazing result that the dimension n at which a simply-connected space X first has nontrivial p-torsion is a stable invariant of the space is established.

Part 5, consisting of five chapters, starts with Chapter 21, which defines the n-th cohomology group of a space X with coefficient group G as $[X, K(G, n)]$ and studies its fundamental properties. Chapter 22 turns to homology. After establishing its fundamental properties, the author investigates multiplicative structures in homology, including a simple Künneth theorem. The relationship between homology and cohomology is discussed. The chapter concludes with the homology and cohomology of H-spaces. Chapter 23 studies cohomology operations, which are natural transformations between cohomology groups. The author works with cohomology theories subject to the wedge and weak equivalence axioms, which enables him to assume, without loss of generality, that all the spaces considered are CW complexes and the cohomology classes are homotopy classes of maps into Eilenberg-MacLane spaces. Chapter 24 is concerned with chain complexes. Although chain complexes emerge as a method of computation and are not central to the definition of the ordinary cohomology or homology of spaces, the fact that cohomology can be computed via chain complexes has a number of very useful consequences, e.g., extensions of the universal coefficients theorem and the Künneth theorem. Chapter 25 is a medley of applications of homology and cohomology.

Part 6, consisting of 8 chapters, begins with Chapter 26, which studies the Wang sequence in detail. Chapter 27 takes the first steps in the systematic study of the question: If we are given a filtration of a space X with known quotients $X_{(s)}/X_{(s-1)}$, what can we learn about the cohomology of X? Chapter 28 is concerned with the Serre filtration of a fibration $p : E \to B$, which gives rise to algebraic filtrations in cohomology and homology, their filtration quotients $\mathrm{Gr}^s H^{s+t}(E)$ being estimated by groups $E_2^{s,t}(p)$. This chapter gives explicit formulas for $E_2^{s,t}(p)$ in terms of the cohomology of the spaces F and B and determines the algebraic structure of the bigraded object $E_2^{*,*}(p)$. Chapter 29 is a short break for the incompressibility of certain maps [*S. Weingram*, "On the incompressibility of certain maps", Ann. Math. (2) 93, 476–485 (1971; Zbl 0214.49904)]. Chapter 30 is concerned with the spectral sequence of a generic filtered space X, namely, an infinite sequence of bigraded chain complexes $E_r^{*,*}(X)$ related to one another by $E_{r+1}^{*,*}(X) = H^{*,*}(E_r^{*,*})$. Chapter 31 is devoted to developing the skill of working with the Leray-Serre spectral sequence. Chapter 32 is concerned with the Bott periodicity theorem [*R. Bott*, "The stable homotopy of the classical groups", Ann. Math. (2) 70, 313–337 (1959; Zbl 0129.15601)] claiming that the Bott map $\beta : \Omega SU \to BU$ is a homotopy equivalence. Chapter 33 works out the cohomology of Eilenberg-MacLane spaces with coefficients in $\mathbb{Z}/p$ at p prime, which is one of the very first triumphs of the spectral sequence technique and allows one to completely determine the Steenrod algebra for ordinary cohomology with coefficients in $\mathbb{Z}/p$.

Part 7, consisting of 4 chapters, begins with Chapter 34, which studies (homotopy) localization of spaces, first in great generality and then focusing on algebraic localizations of spaces. A brief account of rational homotopy theory is given. The chapter concludes with a short treatment of some applications of localization to H-spaces. Chapter 35 is concerned with global properties of the entire graded group $\pi_*(S^n)$, specifically, the p-exponent of $\pi_*(S^n)$. Since working with strongly closed classes and strongly resolving classes rather than with individual spaces greatly simplifies arguments involving the homotopy theory of mapping spaces, Chapter 36 is concerned with these classes of spaces, where the author consistently uses the Serre model structure. Chapter 37 proves a very important theorem of *H. Miller* ["The Sullivan conjecture and homotopical representation theory", Proc. Int. Congr. Math., Berkeley/Calif. 1986, Vol. 1, 580–589 (1987; Zbl 0678.55008)], which is based on a spectral sequence of *A. K. Bousfield* and *D. M. Kan*

[Homotopy limits, completions and localizations. Berlin etc.: Springer-Verlag (1972; Zbl 0259.55004)].

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MSC:

55-01 Textbooks (algebraic topology)
55Pxx Homotopy theory
55Qxx Homotopy groups

Keywords:

homotopy theory; model category; fibration; cofibration; homotopy limit; homotopy colimit; homotopy pullback; homotopy pushout; CW complex; loop space; suspension; Hurewicz map; Borel construction; Serre fibration; Hopf map; Hopf invariant; Lusternik-Schnirelmann category; Mather cube; Leray-Serre spectral sequence; monster; homology; cohomology; pointwise homotopy equivalence; Serre filtration; universal coefficients theorem; J. H. W. Whitehead theorem; Postnikov section; Moore space; Seifert-van Kampen theorem; Künneth theorem; Eilenberg-MacLane space; Wang sequence; Bott periodicity theorem; rational homotopy theory; strongly closed class; strongly resolving class; Serre model structure; Blakers-Massey theorem; Ganea's theorem; Hilton-Milnor theorem; Brown's representability theorem