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**Division algebras and quantum theory.** (English) Zbl 1259.81023

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The orthodox approach to quantum mechanics adopts complex numbers as scalars, while the orthodox one to classical mechanics adopts real numbers as scalars. It is possible to formulate quantum mechanics using real numbers or quaternions instead of complex numbers to a certain extent, though *nonassociative* octonions defy any possibility of formulating quantum mechanics. The pricipal objective in this paper is to show that the nature or God chooses all the three division algebras (in other words, they are merely three aspects of a single unified structure), in sharp contrast to the common view that the nature or God chooses complex numbers in preference to real numbers and quaternions.

The story can be traced back to the early days of group theory, namely, the *Frobenius-Schur indicator* (cf. [Berl. Ber. 1906, 186–208 (1906; JFM 37.0161.01)]). It is equal to 1, 0 or $-1$, which correspond to the irreducible representation $\rho$ of a compact group $G$ on a complex Hilbert space $H$ being real, complex or quaternionic, respectively. It is well known that these three cases can be characterized by considering the dual representation $\rho^*$ on the dual Hilbert space $H^*$, which Freeman Dyson called "threefold way" in [*F. J. Dyson*, J. Math. Phys. 3, 1199–1215 (1962; Zbl 0134.45703)]. The originality of this paper, if any, lies not in mathematics but in physics, more specifically, in its exhaustive exploration of this familiar mathematical fact in connection with the very foundations of quantum theory.

Elementary particles are usually described in irreducible unitary representations of compact groups, so that they appear in three kinds, namely, real, complex and quaternionic. If we take any spin $\frac{1}{2}$-particle with rotational symmetries only, then it is described by a unitary representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$, which is quaternionic. More generally, as far as we are exclusively concerned with representations of $\mathrm{SU}(2)$, particles of half-integer spin are quaternionic, while those of integer spin are real, so that the square of time reversal is 1 for particles of integer spin, while it is $-1$ for particles of half-integer spin. One can see the details of such a story in the paper.

Reviewer: Hirokazu Nishimura (Tsukuba)

**MSC:**

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|---|---|
| 81Q05 | Closed and approximate solutions to quantum-mechanical equations |
| 46S10 | Functional analysis over fields (not $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$) |
| 81R05 | Representations of finite-dimensional groups and algebras in quantum theory |
| 20C35 | Applications of group representations to physics |
| 11R52 | Quaternion and other division algebras: arithmetic, zeta functions |
| 16W10 | Associative rings with involution, etc. |
| 54F05 | Linearly, generalized, and partially ordered topological spaces |



**Keywords:**

division algebra; quantum theory; Jordan algebra; quaternion; octonion; group representation; convex cone; duality

**Full Text:** DOI arXiv

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