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Coarse-graining open Markov processes. (English) Zbl 1406.18004
Theory Appl. Categ. 33, 1223-1268 (2018).

A Markov process is a stochastic model depicting a sequence of transitions between states in which the probability of a transition relies only on the current state. This paper considers only continuous-time Markov processes with a finite set of states. The authors extend coarse-graining, which is a standard method of extracting a simpler Markov process from a more complicated one by identifying states, to open Markov processes, in which probability can flow in or out of certain states called "inputs" and "outputs". Open Markov processes are to be seen as morphisms in a category, where we can compose two open Markov processes by identifying the outputs of the first with the inputs of the second.

The resulting category has been investigated in a number of papers [*J. C. Baez* and *B. Fong*, Theory Appl. Categ. 33, 1158–1222 (2018; Zbl 1402.18005); *B. Fong*, "The algebra of open and interconnected systems", `arXiv:1609.05382`; *B. S. Pollard*, "Open Markov processes and reaction networks", `arXiv:1709.09743`].

This paper goes further to introduce a double category for depiction of coarse-graining. The authors construct a pseudo double category $\mathbb{M}\mathbf{ark}$, in which

1. finite sets as objects,
2. maps between finite sets as vertical 1-morphisms,
3. open Markov processes as horizontal 1-cells (composition of open Markov processes is only weakly associative),
4. morphisms between open Markov processes as 2-morphisms.

In previous work [*J. C. Baez* et al., J. Math. Phys. 57, No. 3, 033301, 30 p. (2016; Zbl 1336.60147); Rev. Math. Phys. 29, No. 9, Article ID 1750028, 41 p. (2017; Zbl 1383.68053)], it was shown that black-boxing is a symmetric monoidal functor, where Markov processes are defined to be a directed multigraph, while this paper works directly with the Hamiltonians, skipping the directed multigraphs.

The principal result in this paper is that black-boxing gives a symmetric monoidal double functor from the double category $\mathbb{M}\mathbf{ark}$ to another double category $\mathbb{L}\mathbf{inRel}$, which has

1. finite-dimensional real vector spaces as objects,
2. linear maps as vertical 1-morphisms,
3. linear relation as horizontal 1-cells,
4. squares
$$\begin{tikzcd} V_1 \arrow[r, "R\subseteq V_1\oplus V_2"] \arrow[d, "f"'] & V_2 \arrow[d, "g"] \\ W_1 \arrow[r, "S\subseteq W_1\oplus W_2"] & W_2 \end{tikzcd}$$
with $(f\oplus g)R\subseteq S$ as 2-morphisms.

The most fastidious part in its proof is to show that black-boxing a composite of open Markov processes gives the composite of their black-boxings, that is to say, that black-boxing preserves composition of horizontal 1-cells, which was fortunately established by simply adapting a previous argument [Zbl 1383.68053]. In the final section (§6), the authors construct symmetric monoidal bicategories **Mark** and **LinRel** from the symmetric monoidal double categories $\mathbb{M}\mathbf{ark}$ and $\mathbb{L}\mathbf{inRel}$ by using a result of [*M. A. Shulman*, "Constructing symmetric monoidal bicategories", `arXiv:1004.0993`].

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MSC:

18D05 Double categories, 2-categories, bicategories and generalizations
18D10 Monoidal, symmetric monoidal and braided categories
60J27 Continuous-time Markov processes on discrete state spaces
16B50 Category-theoretic methods and results in associative algebra

Keywords:

double category; cospan; Markov process; coarse-graining; network; black box

Full Text: Link

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