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From Stein to Weinstein and back. Symplectic geometry of affine complex manifolds. (English) Zbl 1262.32026

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Stein manifolds have their intrinsic symplectic geometry, which is responsible for many interesting phenomena in complex geometry and analysis. The principal objective in this monograph is a systematic exploration of this symplectic geometry (transition from Stein to Weinstein). The monograph is engaged in the classification of Stein structures up to deformation, but not up to biholomorphism. It is well known that the classification of Stein structures up to biholomorphisms is highly elusive, see, e.g., [*S. G. Krantz*, Function theory of several complex variables. Providence, RI: American Mathematical Society (AMS), AMS Chelsea Publishing (2001; Zbl 1087.32001)]. Since there is a natural path from Weinstein to Stein, complex-geometric questions about Stein manifolds are reduced to symplecto-geometric ones about Weinstein manifolds.

The authors develop techniques for answering the latter questions. It is not difficult to point out two necessary conditions for the existence of a Weinstein structure on a given smooth manifold of dimension $2n$. It was established by *J. Eliashberg* [Int. J. Math. 1, No. 1, 29–46 (1990; Zbl 0699.58002)] that these two conditions are sufficient as far as $2n \neq 4$. The result was refined and extended in [*L. Aizenberg*, Stud. Math. 180, No. 2, 161–168 (2007; Zbl 1118.32001); *R. E. Gompf*, CRM Proceedings and Lecture Notes 49, 229–249 (2009; Zbl 1193.32012); *F. Forstnerič* and *M. Slapar*, Math. Res. Lett. 14, No. 2, 343–357 (2007; Zbl 1134.32007)] and so on. The monograph gives other refinements and extensions. The situation is drastically different in case of $2n = 4$, see, e.g., [*P. Lisca* and *G. Matić*, Invent. Math. 129, No. 3, 509–525 (1997; Zbl 0882.57008)]. Therefore it is intriguing to note the topological analogue in [*R. E. Gompf*, Ann. Math. (2) 148, No. 2, 619–693 (1998; Zbl 0919.57012)], which uses the technique of Casson handles as well as [Zbl 0882.57008)] based on Seiberg-Witten theory, and which is widely beyond the scope of the present monograph.

The monograph consists of 5 parts, one beginning chapter devoted to an overview, and three apendices dealing with some algebraic topology, obstructions to formal Legendrian isotopies and biographical notes of some outstanding pioneers such as Oka, Grauert, Gromov and Weinstein.

The original purpose of the monograph was a complete and detailed exposition of the existence theorem for Stein structures in [Zbl 0699.58002], which is achieved in Parts 1 and 2 composed of 7 chapters. Chapters 2 and 3 explore basic properties and examples of J-convex functions and hypersurfaces, establishing *R. Richberg*'s theorem [Math. Ann. 175, 257–286 (1968; Zbl 0153.15401)] in particular. Chapter 4 constructs special hypersurfaces, which play a crucial role in extending J-convex functions over handles. Chapter 5 reviews the notion of holomorphic convexity and its generalization to J-convexity, Grauert's Oka principle and its applications, the holomorphic filling problem for J-convex CR manifolds, and real analytic approximations. Chapter 6 collects some relevant facts from symplectic and contact geometry. Chapter 7, consisting of 9 sections, collects some relevant h-principles mostly from [*M. Gromov*, Partial differential relations. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 9. Berlin etc.: Springer-Verlag. (1986; Zbl 0651.53001)] and [*Y. Eliashberg* and *N. Mishachëv*, Introduction to the h-principle. Mathematical Surveys and Monographs. 48. Providence, RI: American Mathematical Society (AMS). (2002; Zbl 1008.58001)] in the first four sections. The suceeding three sections are concerned with Legendrian embeddings. The concluding two sections combine h-principles for totally real embeddings with those for isotropic contact embeddings to obtain h-principles for totally real discs attached to J-convex boundaries. Chapter 8 establishes Theorem 1.5 claiming the existence of Stein structures.

Part 3, consisting of two chapters, is concerned with the Morse-Smale theory for J-convex functions. Chapter 9 reviews the Morse-Smale theory and the h-cobordism theorem, discussing basic facts concerning gradient-like vector fields and recalling the "two-index theorem" in [*A. E. Hatcher* and *J. B. Wagoner*, Astérisque 6, 8–238 (1973; Zbl 0274.57010)]. Chapter 10, depending on the techniques developed in Chapters 3 and 4, discusses modifications of J-convex Morse functions on a given complex manifold.

Part 4, consisting of 5 chapters, bears the title of the monograph. Chapter 11 introduces Weinstein cobordisms and manifolds and establish their basic properties. Chapter 12 carries over various constructions for Morse cobordisms discussed in Chapter 9 to Weinstein cobordisms, establishing the easier Weinstein analogues of the modifications of Stein structures studied in Chapter 10. Chapter 13 proves a more precise version of Theorem 1.5 by splitting it into two theorems, namely, the theorem on the existence of Weinstein structures and the theorem on upgrading a Weinstein structure to a Stein structure. Chapter 14 is devoted to showing that flexible Weinstein structures in dimension > 4 are indeed flexible. That is to say, any Morse homotopy can be followed by a flexible Weinstein homotopy, and two flexible Weinstein structures on the same manifold whose symplectic forms are homotopic as nondegenerate 2-forms are Weinstein homotopic. Chapter 15 demonstrates that Weinstein homotopies can be lifted to Stein homotopies. Chapters 14 and 15 are the very core of the monograph.

Part 5 consists of two chapters. Chapter 16 concerns the situation in dimension 2. The results in the chapter are based upon foliations by holomorphic discs introduced by *Y. Eliashberg* [Lect. Note Ser. 151, 45–72 (1990; Zbl 0731.53036)]. In Chapter 17 the authors discuss how to distinguish Stein and Weinstein structures up to deformation equivalence, the main tool for which is symplectic homology. By considering their underlying Liouville structures, symplectic homology gives rise to deformation invariants of Weinstein structures.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

32Q28	Stein manifolds
53D35	Global theory of symplectic and contact manifolds
53-02	Research monographs (differential geometry)

Cited in **7** Reviews
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