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Sheaves of algebras over Boolean spaces. (English) Zbl 1243.06001
Basel: Birkhäuser (ISBN 978-0-8176-4218-1/hbk; 978-0-8176-4642-4/ebook). xii, 329 p. (2012).

The reductionist philosophy has a long and intricate history in algebra. Its broad motivation has been to break up a complicated algebra into simpler pieces. The obvious decomposition to try first is a direct product. The celebrated Wedderburn-Artin theorem, claiming that an Artinian semisimple ring is isomorphic essentially uniquely to a product of finitely many $n \times n$ matrix rings over a division ring, is of this type. Being a bit more generous, we have the representation as a subdirect product, which goes back to *M. H. Stone* ["The theory of representations for Boolean algebras", Trans. Am. Math. Soc. 40, 37–111 (1936; Zbl 0014.34002)] and *G. Birkhoff* ["Subdirect unions in universal algebra", Bull. Am. Math. Soc. 50, 764–768 (1944; Zbl 0060.05809)]. The main drawback of subdirect products is that, while factors may be commonplace and well understood, the transfer of an argument from the components to the whole algebra may fail. Thus, one grafts topological spaces, particularly Boolean spaces, onto subdirect products so that elements of the subdirect product become continuous functions. The author argues that, by using what he dubs shells, one can generalize well beyond ring theory a number of classical results on biregularity, strong regularity, and lack of nilpotents. This monograph adapts the intuitive idea of a metric space to universal algebra, leading to the useful device of a complex from which a sheaf is constructed directly. The gist of the author's ideas is that one need not look at all congruences of an algebra, but at only some of them comprising a Boolean subsemilattice of congruences and, more typically, at others splitting the algebra into a product of complementary factors. As for prerequisites, the reader should have a nodding acquaintance with universal algebra, logic, categories, topology, and Boolean algebra.

The book consists of twelve chapters. The organization of the book is as follows.

After an introduction in Chapter I, Chapter II lays out the traditional background from general algebra.

Chapter III outlines the concepts and theorems needed from several disciplines, namely, from equational logic, category theory, topology, and Boolean algebras, including Stone's representation theorem.

Chapter IV sets the stage for the book proper by introducing the notion of a complex and showing that it always gives a sheaf of algebras. The constructions in this chapter are finally epitomized as an adjoint situation between the categories $\mathfrak{Complex}$ and $\mathfrak{Sheaf}$ for a given algebraic type.

Chapter V establishes a theorem stating that any Boolean subsemilattics of an algebra determines a complex over a Boolean space, which in turn determines a sheaf of algebras over the same space. The original algebra is a subalgebra of the algebra of all global sections of a sheaf of quotient algebras. The converse, that every representation of an algebra by a sheaf of algebras over a Boolean space must arise by the previous construction from some Boolean subsemilattice of congruences is also established. The last section reformulates the results categorically. It is shown that the category $\mathfrak{BooleBraceRed}$ of reduced Boolean braces forms an adjunction with the category $\mathfrak{CompBooleRed}$ of reduced complexes over Boolean spaces as well as the category $\mathfrak{SheafBooleRed}$ of reduced sheaves over Boolean spaces.

Speaking from a coign of vantage of category theory, Chapter VI demonstrates that $\mathfrak{FactorBraceRed}$ is categorically equivalent to $\mathfrak{SheafBooleRed}$, while $\mathfrak{AlgBFC}$ is equivalent to $\mathfrak{SheafBooleRedFt}$.

From the viewpoint of category theory, Chapter VII is devoted to a theorem asserting that the category of shells with conformal homomorphisms is equivalent to the category of reduced sheaves over Boolean spaces.

One of the highlights of this monograph is the generalization of unital shells of [*J. Kist*, "Compact spaces of minimal prime ideals", Math. Z. 111, 151–158 (1969; Zbl 0177.06404)] to the decomposition of commutative Baer unital rings into a sheaf of integral domains over a Boolean space, which in turn is a generalization of the classical result that every von Neumann regular commutative unital ring is a subdirect product of fields.

Chapter VIII gives some applications. The first main result is that the category of Baer-Stone two-sided unital shells with conformal homomorphisms is categorically equivalent to the category of sheaves of integral shells. The second main result is that a biregular unital half-shell is isomorphic to the half-shell coming from a reduced and factor-transparent sheaf over a Boolean space with simple stacks, provided certain technical conditions hold. This is a generalization of [*R. F. Arens* and *I. Kaplansky*, "Topological representation of algebras", Trans. Am. Math. Soc. 63, 457–481 (1948; Zbl 0032.00702); *J. Dauns* and *K. H. Hofmann*, "The representation of biregular rings by sheaves", Math. Z. 91, 103–123 (1966; Zbl 0178.37003)].

The sheaf representations of Chapters VII and VIII are all surjective. Relaxing this, Chapter IX establishes some theorems claiming that an algebra may be represented merely as some subalgebra of the algebra of all global sections of a sheaf.

Chapter X tests the theory of the previous chapters to see how readily one may find sheaf representations of algebras in the varieties generated by preprimal algebras.

Getting away from the language of shells and half-shells and returning to general algebras, Chapter XI consists of two independent sections. The first section, generalizing [*W. D. Burgess* and *W. Stephenson*, "An analogue of the Pierce sheaf for non-commutative rings", Commun. Algebra 6, 863–886 (1978; Zbl 0374.16017)], shows how the sheaf representation in Chapter VI may be iterated until all the stacks become directly indecomposable. Observing that $\mathbf{Con}\,A$ is a shell, the second section bootstraps the main result of Chapter VII to any algebra A of any type.

Chapter XII depicts five broad landscapes into which the interested reader may venture. The first is classical algebra. The second is the invention of algebras capturing various logics. The third section is concerned with model theory, dealing with preservation of properties, decidability of first-order theories, and model completeness. The fourth section gives three loosely connected topics, namely, generalized metrics, the weaking of Boolean algebras, and the mixing of duality theory with sheaf theory. The fifth section is concerned with the fact that the same algebra is often represented by several different sheaves, treating bounded distributive lattices, unitary rings, and biregular rings briefly by way of example and giving four representations of Boolean rings.

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MSC:

06-02	Research monographs (ordered structures)	Cited in **2** Documents
06E15	Stone spaces and related constructions	
06E20	Ring theoretic properties (Boolean algebras)	
08-02	Research monographs (general algebraic systems)	
18-02	Research monographs (category theory)	
18F20	Categorical methods in sheaf theory	

Keywords:

Boolean space; subdirect product; sheaf; shell; half-shell; complex; congruence; Boolean subsemilattice; brace; band

Full Text: DOI