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A prehistory of n-categorical physics. (English) Zbl 1236.81006

Halvorson, Hans (ed.) et al., Deep beauty. Understanding the quantum world through mathematical innovation. Papers based on the presentations at the deep beauty symposium, Princeton, NJ, USA, October 3–4, 2007. Cambridge: Cambridge University Press (ISBN 978-1-107-00570-9/hbk; 978-1-139-06608-2/ebook). 13-128 (2011).

As the authors have stated, this paper "is a highly subjective chronology describing how physicists have begun to use ideas from n-category theory in their work, often without making this explicit." They have begun with the discovery of special relativity and quantum mechanics, ending at the dawn of the twenty-first century. Since the developments in the twenty-first century are so thick that the authors have given up putting them in their proper perspective.

The chronology begins with a citation from *J. C. Maxwell* [Matter and motion. With notes and appendices by Joseph Larmor. Reprint of the 1876 original. Reprint of the 1876 original. Amherst, NY: Prometheus Books (2002; Zbl 1060.01011)]. The second is *H. Poincaré*'s fundamental groups [J. de l'Éc. Pol. (2) I. 1–123 (1895; JFM 26.0541.07)] hinting at the unification of space and symmetry, which was later to become one of the main themes of n-category theory. The third is *H. A. Lorentz* transformations [Leiden. E. J. Brill. 139 S. 8° (1895; JFM 26.1032.06); ibid. 138 S. 8° (1895; JFM 25.1632.01); Amst. Ak. Versl. 12, 986–1009 (1904; JFM 35.0837.03)]. The fourth is again *H. Poincaré* [C. R. Acad. Sci., Paris 140, 1504–1508 (1905; JFM 36.0911.02)] as a keyman in the formation of special relativity. The fifth is *A. Einstein* [Ann. der Phys. (4) 17, 891–921 (1905; JFM 36.0920.02)] surely as an inventor of special relativity. The sixth is *H. Minkowski* [Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. 1908, 53–111 (1908; JFM 39.0909.02)] for his Minkowski spacetime. The succeeding two are *W. Heisenberg* [Z. Phys. 33, 879–893 (1925; JFM 51.0728.07)] and *M. Born* [Z. Phys. 37, 863–867 (1926; JFM 52.0973.03)]. It seems that the authors have made an error in attributing Born's discovery of the interpretation of $\psi^*\psi$ in the Schrödinger equation as the probability density function, for which he won his Nobel prize, not to 1926, when Schrödinger found out his famous equation to be called the Schrödinger equation, but mysteriously to 1928. It is well known that Einstein was against this interpretation, saying that God does not play dice. The ninth and the tenth are *J. von Neumann* [Mathematische Grundlagen der Quantentheorie. Berlin: Julius Springer (1932; Zbl 0005.09104)] and *E. P. Wigner* [Ann. Math. (2) 40, 149–204 (1939; Zbl 0020.29601)], respectively. The eleventh is [*S. Eilenberg* and *S. MacLane*, Trans. Am. Math. Soc. 58, 231–294 (1945; Zbl 0061.09204)], inventing the notion of a category. The twelfth is Feynman's famous lecture in a small conference at Shelter Island in 1947, which surpassed the understanding of most physicists except Schwinger. The reviewer would like to mention [*A. Wüthrich*, The genesis of Feynman diagrams. Dordrecht: Springer (2010; Zbl 1226.81008)] as a good reference here. The thirteenth is *C. N. Yang* and *R. Mills*' generalization of Maxwell's equations ["Conservation of isotopic spin and isotopic gauge invariance", Phys. Rev. 96, No. 1, 191–195 (1954; `doi:10.1103/PhysRev.96.191`)], which is now known as the Yang-Mills theory. The fourteenth is *S. MacLane*'s introduction of monoidal categories [Rice Univ. Stud. 49, No. 4, 28–46 (1963; Zbl 0244.18008)]. The fifteenth is *F. W. Lawvere*'s thesis on functorial semantics [Repr. Theory Appl. Categ. 2004, No. 5, 1–121 (2004; Zbl 1062.18004)], whose impact is apparent in [*J. M. Boardman* and *R. M. Vogt*, Homotopy invariant algebraic structures on topological spaces. York: Springer-Verlag (1973; Zbl 0285.55012); *S. MacLane*, Bull. Am. Math. Soc. 71, 40–106 (1965; Zbl 0161.01601); *J. P. May*, The geometry of iterated loop spaces. York: Springer-Verlag (1972; Zbl 0244.55009)] and so on, reaching "the definitions of conformal field theory and topological quantum field theory propounded by Segal and Atiyah in the late 1980s. "The sixteenth is *J. Bénabou* et al.'s introduction of bicategories [Reports of the Midwest Category Seminar. York: Springer-Verlag (1967; Zbl 0165.33001)]. The seventeenth is *R. Penrose* [in: Combinat. Math. Appl., Proc. Conf. Math. Inst., Oxford 1969, 221–244 (1971; Zbl 0216.43502)]. The eighteenth is [*G. Ponzano* and *T. Regge*, in: F. Bloch et al., Spectroscopic and group theoretical methods in physics. Amsterdam: North-Holland Publishing Company. 75–103 (1968; Zbl 0172.27401)], who "applied Penrose's theory of spin networks before it was invented to relate tetrahedron-shaped spin networks to gravity in three-dimensional spacetime. The nineteenth is Grothendieck's "Pursuing Stacks", a very long letter to D. Quillen in 1983, in which he "fantasized about n-categories for higher n – even $n = \infty$ – and their relation to homotopy theory.” The twentieth is string theory which experienced the outburst in the 1980s. The reviewer would like to mention [*R. Blumenhagen* et al., Basic concepts of string theory. Berlin: Springer (2012; Zbl 1262.81001)] as a recommendable textbook on the subject, though the great influence of [*M. B. Green* et al., Superstring theory. Volume 1: Introduction. Volume 2: Loop amplitudes, anomalies and phenomenology. Cambridge etc.: Cambridge University Press (1987; Zbl 0619.53002)] on the above textbook is apparent. Just as essentially one-dimensional Feynman diagrams in quantum field theory are replaced by two-dimensional diagrams depicting string worldsheets in string theory, the mathematics of categories should be replaced by that of bicategories. The twenty-first is [*A. Joyal* and *R. Street*, Adv. Math. 102, No. 1, 20–78 (1993; Zbl 0817.18007)]. The twenty-second is [*V. F. R. Jones*, Bull. Am. Math. Soc., New Ser. 12, 103–111 (1985; Zbl 0564.57006)]. “The work of Jones led researchers toward a wealth of fascinating connections between von Neumann algebras, higher categories, and quantum field theory in two- and three-dimensional spacetime.” The twenty-third is [*P. Freyd* et al., Bull. Am. Math. Soc., New Ser. 12, 239–246 (1985; Zbl 0572.57002)] and [Zbl 0638.57003]. The twenty-fourth is [*V. G. Drinfel'd*, in: Proc. Int. Congr. Math., Berkeley/Calif. 1986, Vol. 1, 798–820 (1987; Zbl 0667.16003)], which is “the culmination of a long line of work on exactly solvable problems in low-dimensional physics”. The twenty-fifth is [*G. B. Segal*, in: Differential geometrical methods in theoretical physics, Proc. 16th Int. Conf., NATO Adv. Res. Workshop, Como/Italy 1987, NATO ASI Ser., Ser. C 250, 165–171 (1988; Zbl 0657.53060)], which proposed axioms describing a conformal field theory. The twenty-sixth is [*M. Atiyah*, Publ. Math., Inst. Hautes Étud. Sci. 68, 175–186 (1988; Zbl 0692.53053)], whose goal was to formalize [*E. Witten*, Commun. Math. Phys. 117, No. 3, 353–386 (1988; Zbl 0656.53078)]. Ironically, “these invariants have led to a revolution in our understanding of four-dimensional topology”, while it has never successfully dealt with the Donaldson theory. The twenty-seventh is *R. H. Dijkgraaf*' purely algebraic characterization of two-dimensional topological quantum field theories in terms of commutative Frobenius algebras in his [A geometric approach to two-dimensional conformal field theory. Utrecht, NL: University of Utrecht (PhD Thesis) (1989)]. The twenty-eighth is [*S. Doplicher* and *J. E. Roberts*, Invent. Math. 98, No. 1, 157–218 (1989; Zbl 0691.22002)], which shows “one could start with a fairly general quantum field theory and compute its gauge group instead of putting the group in by hand”. The twenty-ninth is [*N. Yu. Reshetikhin* and *V. G. Turaev*, Commun. Math. Phys. 127, No. 1, 1–26 (1990; Zbl 0768.57003)], where the author summarizes a bit of the theory of quantum groups in its modern form. The thirtieth is [*E. Witten*, Adv. Ser. Math. Phys. 9, 239–329 (1989; Zbl 0726.57010); ibid. 17, 361–451 (1994; Zbl 0818.57014)], which gave an intrinsically three-dimensional description of the Jones polynomial by using a gauge field theory in three-dimensional spacetime called the Chern-Simons theory. The thirty-first is loop quantum gravity initiated by *C. Rovelli* and *L. Smolin* in [“Loop space representation of quantum general relativity”, Nucl. Phys. B 331, No. 1, 80–152 (1990; `doi:10.1016/0550-3213(90)90019-A`)]. The thirty-second is [*M. Kashiwara*, Duke Math. J. 63, No. 2, 465–516 (1991; Zbl 0739.17005); ibid. 69, No. 2, 455–485 (1993; Zbl 0774.17018); Commun. Math. Phys. 133, No. 2, 249–260 (1990; Zbl 0724.17009)] and [*G. Lusztig*, J. Am. Math. Soc. 3, No. 2, 447–498 (1990; Zbl 0703.17008); ibid. 4, No. 2, 365–421 (1991; Zbl 0738.17011); *I. Grojnowski* and *G. Lusztig*, Contemp. Math. 153, 11–19 (1993; Zbl 1009.17502)] related with canonical bases. Their appearance hints “that quantum groups are just shadows of more interesting structures where the canonical basis elements become objects of a category, multiplication becomes the tensor product in this category, and addition becomes a direct sum in this category”, which might be called a categorified quantum group. The thirty-third is [*M. M. Kapranov* and *V. A. Voevodsky*, Proc. Symp. Pure Math. 56, 177–259 (1994; Zbl 0809.18006)], initiating 2-vector spaces and what are now called braided monoidal bicategories. The authors of the paper argued that, just as any solution of the Yang-Baxter equation gives a braided monoidal category, any solution of the Zamolodchikov tetrahedron equation gives a braided monoidal bicategory. The thirty-fourth is [*N. Reshetikhin* and *V. G. Turaev*, Invent. Math. 103, No. 3, 547–597 (1991; Zbl 0725.57007)], constructing invariants of 3-manifolds from quantum groups, which were later seen to be part of a full-fledged three-dimensional topological quantum field theory known as the Witten-Reshetikhin-Turaev theory. The thirty-fifth is [*V. G. Turaev* and *O. Y. Viro*, Topology 31, No. 4, 865–902 (1992; Zbl 0779.57009)], constructing another invariant of 3-manifolds from the modular category stemming from quantum SU(2). It is now known as a part of a full-fledged three-dimensional topological quantum field theory, and its relation with the Witten-Reshetikhin-Turaev theory “subtle and interesting.” The thirty-sixth is [*M. Fukuma* et al., Commun. Math. Phys. 161, No. 1, 157–175 (1994; Zbl 0797.57012)], which “found a way to construct two-dimensional topological quantum field theories from semisimple algebras”. The authors “essentially created a recipe to turn any two-dimensional cobordism into a string diagram” and, with a little extra work, it gives a topological quantum field theory called Z. The thirty-seventh is [*J. W. Barrett* and *B. W. Westbury*, Trans. Am. Math. Soc. 348, No. 10, 3997–4022 (1996; Zbl 0865.57013)], which can be seen as “a categorical version of the Fukuma-Hosono-

Kawai construction". The thirty-eighth is [*V. G. Turaev*, J. Differ. Geom. 36, No. 1, 35–74 (1992; Zbl 0773.57012)], followed by [*V. G. Turaev*, Quantum invariants of knots and 3-manifolds. Berlin: Walter de Gruyter (1994; Zbl 0812.57003)], which "amounts to a four-dimensional analogue of the Traev-Viro-Barrett-Westbury construction". The thirty-ninth is famous [*V. G. Turaev*, Quantum invariants of knots and 3-manifolds. Berlin: Walter de Gruyter (1994; Zbl 0812.57003)] arriving at a deeper understanding of quantum groups, which was based on ideas of Witten. The fortieth is [*R. J. Lawrence*, in: Quantum topology. Based on an AMS special session on topological quantum field theory, held in Dayton, OH, USA on October 30-November 1, 1992 at the general meeting of the American Mathematical Society. Singapore: World Scientific. 191–208 (1993; Zbl 0839.57017)] followed by [*R. J. Lawrence*, J. Pure Appl. Algebra 100, No. 1–3, 43–72 (1995; Zbl 0827.57011)]. "The essential point is that we can build any n-dimensional spacetime out of a few standard building blocks, which can be glued together locally in a few standard ways". The forty-first is [*L. Crane* and *I. B. Frenkel*, J. Math. Phys. 35, No. 10, 5136–5154 (1994; Zbl 0892.57014)] discussing "algebraic structures that provide topological quantum field theories in various low dimensions". The forty-second is [*D. S. Freed*, Commun. Math. Phys. 159, No. 2, 343–398 (1994; Zbl 0790.58007)] exhibiting "how higher-dimensional algebraic structures arise naturally from the Lagrangian formulation of topological quantum field theory". The forty-third is [*M. Kontsevich*, in: Proceedings of the international congress of mathematicians, ICM '94, August 3-11, 1994, Zürich, Switzerland. Vol. I. Basel: Birkhäuser. 120–139 (1995; Zbl 0846.53021)], "which led to a burst of work relating string theory to higher categorical structures". The forty-fourth is [*R. Gordon* et al., Mem. Am. Math. Soc. 558, 81 p. (1995; Zbl 0836.18001)], which "is a precise working out of the categorical Eckman-Hilton argument". Now we have finally come to [*J. C. Baez* and *J. Dolan*, J. Math. Phys. 36, No. 11, 6073–6105 (1995; Zbl 0863.18004)] as the forty-fifth. Its key part is the periodic table of n-categories. The last (the forty-sixth) is [*M. Khovanov*, Duke Math. J. 101, No. 3, 359–426 (2000; Zbl 0960.57005)].

The authors have stated that they are scientists rather than historians of science, so they were trying to make a specific scientific point rather than accurately describe every twist and turn in a complex sequence of events. But the reviewer should say regrettably that the authors have not made due effort to be exact and adequate in references and citations even as scientists. Nevertheless, the paper marvelously succeeded in following the threads of various ideas and intertwining them. The story that the authors narrate is highly interesting and even thrilling. Besides, the authors have tried every effort to make the paper more accessible by including a general introduction to n-categories. Applications of n-categories started only around the 1980s, particularly in string theory and spin foam models of loop quantum gravity. It should be stressed again and again that these physical theories are spectulative at present and are not ready for experimental tests at all. The history of physics is full of theories resulting in fiasco (e.g., Kaluza-Klein theory). Therefore the authors have modestly and wisely chosen to speak of a prehistory in place of a history.

For the entire collection see [Zbl 1222.81033].

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

81P05 General and philosophical topics in quantum theory
81-03 Historical (quantum theory)
00A79 Physics
01A60 Mathematics in the 20th century
01A61 Mathematics in the 21st century

Cited in **3** Documents

Keywords:

n-category; loop quantum gravity; string theory; quantum theory; Feynman diagram; category theory; topological quantum field theory; electromagnetism; monoidal category; functorial semantics; Yang-Mills theory; spin network; conformal field theory; bicategory; Jones polynomial; low-dimensional topology; Donaldson theory; quantum group; canonical basis; braided monoidal bicategory; semisimple algebra; Eckman-Hilton argument; Frobenius algebra

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