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Gauge theory and topology. (Geiji riron to toporoji.) (Japanese) Zbl 1369.81003
Tokyo: Maruzen (ISBN 978-4-621-06457-3). 449 p. (2012).

Gauge theory as a branch of mathematics has various ramifications. This book, consisting of 6 chapters and an appendix, is concerned exclusively with its aspects related to Yang-Mills fields. Gauge theory of Yang-Mills fields is now engaged mainly in the study of their moduli space by methods of functional analysis and its applications to 4-dimensional topology. Yang-Mills fields began to draw attentions of mathematicians, particularly, geometers in the 1970s [*M. F. Atiyah* et al., Proc. R. Soc. Lond., Ser. A 362, 425–461 (1978; Zbl 0389.53011); Philos. Trans. R. Soc. Lond., Ser. A 308, 523–615 (1983; Zbl 0509.14014); Commun. Math. Phys. 61, 97–118 (1978; Zbl 0387.55009); Phys. Lett., A 65, No. 3, 185–187 (1978; Zbl 0424.14004)]. Gauge theory as a branch of mathematics debuted as a nonlinearization of the theory of harmonic integrals, in which the theory of elliptic partial differential equations found its generalizations and applications. On the other hand, the study of Yang-Mills equations by methods of functional analysis was pioneered by Uhlenbeck and Taubes. In the 1980s, *S. K. Donaldson* established a path from gauge theory to 4-dimensional topology [J. Differ. Geom. 18, 279–315 (1983; Zbl 0507.57010); Proc. Lond. Math. Soc. (3) 50, 1–26 (1985; Zbl 0529.53018); J. Differ. Geom. 24, 275–341 (1986; Zbl 0635.57007); J. Differ. Geom. 26, 397–428 (1987; Zbl 0683.57005); J. Differ. Geom. 26, 141–168 (1987; Zbl 0631.57010); Topology 29, No. 3, 257–315 (1990; Zbl 0715.57007)]. Topological field theory stemmed from the intimate relationship between Donaldson invariants and Floer homology based on Yang-Mills fields [*A. Floer*, Commun. Math. Phys. 118, No. 2, 215–240 (1988; Zbl 0684.53027); Lond. Math. Soc. Lect. Note Ser. 150, 97–114 (1990; Zbl 0788.57008)]. One can find a cosmorama of these flows of ideas in a single volume.

Chapter 0 is a prelude. §0.1 is a brief but readable introduction. The author's platform is that it is the arena of gauge theory that can resurrect a flamboyant combination of various areas of modern mathematics in the middle of the previous century when differential topology emerged [*R. Thom*, C. R. Acad. Sci., Paris 236, 1733–1735 (1953; Zbl 0050.39602); Colloque de Topologie de Strasbourg 1952, No. 7, 4 S. (1953; Zbl 0053.30103); *S. Smale*, Ann. Math. (2) 74, 391–406 (1961; Zbl 0099.39202); Bull. Am. Math. Soc. 66, 373–375 (1960; Zbl 0099.39201); *J. W. Milnor*, Ann. Math. (2) 64, 399–405 (1956; Zbl 0072.18402); Bull. Soc. Math. Fr. 87, 439–444 (1959; Zbl 0096.17801); Am. J. Math. 81, 962–972 (1959; Zbl 0111.35501)]. §0.2 is concerned with past and present of Morse theory. This section consists of 6 subsections: (I) The beginning of Morse theory (II) Morse inequality (III) The cancellation of critical points and the Poincaré conjecture (IV) Intersection resolutions and the h-cobordism theorem (V) Infinite-dimensional Morse theory (VI) Harmonic mappings and bubbles. §0.3 deals with a transition from the de Rham theorem to the index theorem of elliptic operators. This section consists of 5 subsections: (I) The de Rham theorem (II) The theory of harmonic integrals (III) The Gauss-Bonnet theorem (IV) Vector bundles (V) Elliptic operators. §0.4 gives a sketch of 4-dimensional topology. This section consists of 10 subsections: (I) The Rokhlin theorem (II) Symmetric quadratic forms with integer coefficients (III) $K3$ surfaces (IV) Surgery theory in 4-dimensions (V) The classification of 4-dimensional topological manifolds (VI) Complex surfaces 1 (blowing up) (VII) Complex surfaces 2 (pluricanonical mappings) (VIII) Complex surfaces 3 (rational surfaces) (IX) Complex surfaces 4 ($K3$ surfaces) (X) Complex surfaces 5 (elliptic surfaces).

Chapter 1 is a highly standard course on vector bundles and connections. §1.1 deals with connections and gauge transformations. §1.2 is concerned with curvatures and characteristic classes. §1.3 explains spin structures and Dirac operators.

Chapter 2 is concerned with an outline of gauge theory. §2.1 suggests how to study properties of 4-dimensional manifolds by investigating the moduli space of anti-self-dual (usually abbreviated to ASD) connections. The succeeding sections explain some properties of the above moduli space. §2.2 deals with (I) the index theorem and dimensions and (II) reducible connections and singularities. §2.3 is devoted to the index theorem of families of elliptic operators. §2.4 discusses the existence of isolated waves of the ASD equation.

Chapter 3 is devoted to the analytical aspects of ASD equations in order to realize the geometric ideas of Chapter 2. §3.1 is a brief survey of Sobolev spaces. §3.2 explains how to introduce coordinates in the moduli space $\mathcal{M}(M,\zeta)$. §3.3 is concerned with the so-called Uhlenbeck principle [*K. K. Uhlenbeck*, Bull. Am. Math. Soc., New Ser. 1, 579–581 (1979; Zbl 0416.35026); Commun. Math. Phys. 83, 11–29 (1982; Zbl 0491.58032); ibid. 83, 31–42 (1982; Zbl 0499.58019)]. §3.4 deals with transversal regularity in detail, which is crucial in establishing that the moduli space of ASD connections is a manifold. §3.5 completes the definition of Donaldson invariants by using the results of the previous sections of this chapter.

In Chapter 4 the author extends the theory of Donaldson polynomials to manifolds with boundary. §4.1 is devoted to topological field theory, explaining Atiyah's axiom. §4.2 deals with an outline of Floer homology. §4.3 is concerned with relative Donaldson polynomials. The remaining two sections are devoted to proofs of several theorems claimed in the previous sections. §4.4 deals with elliptic operators on non-compact manifolds. §4.5 studies a kind of nonlinealization of the preceding section.

In Chapter 5 the author gives an application of the general theory of the moduli space of ASD connections after [Duke Math. J. 64, No. 2, 229–241 (1991; Zbl 0754.57015)], which presupposes no algebraic geometry. It settles negatively Kodaira's problem whether any complex surface obtained from a $K3$ surface by the logarithmic transformation is diffeomorphic to a $K3$ surface or not [*K. Kodaira*, in: Essays Topol. Relat. Top., Mem. dedies a Georges de Rham, 58–69 (1970; Zbl 0212.28403)]. Kronheimer's method does not use polynomial invariants through SU(2) bundles but invariants through SO(3) bundles, which are easier to treat in various senses.

The appendix deals with a glimpse into Seiberg-Witten theory.

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MSC:

81-02	Research monographs (quantum theory)
81T13	Yang-Mills and other gauge theories
53C07	Special connections and metrics on vector bundles (Hermite-Einstein-Yang-Mills)
53Z05	Applications of differential geometry to physics
58E15	Applications of variational methods to extremal problems in several variables; Yang-Mills functionals
14D21	Applications of vector bundles and moduli spaces in mathematical physics
70S15	Yang-Mills and other gauge theories