Hofmann, Dirk (ed.); **Seal, Gavin J. (ed.)**; **Tholen, Walter (ed.)**

Monoidal topology. A categorical approach to order, metric, and topology. (English)

Zbl 1297.18001

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The search for a satisfactory notion of convergence has been one of the major interests in topology from scratch: *M. Fréchet* [C. R. Acad. Sci., Paris 139, 848–850 (1905; JFM 35.0389.02); Rend. Circ. Mat. Palermo 22, 1–74 (1906; JFM 37.0348.02)] introduced metric spaces and considered sequential convergence in an abstract manner. *E. H. Moore* [Nat. Acad. Proc. 1, 628–632 (1915; JFM 45.0426.03)] and *E. H. Moore* and *H. L. Smith* [Am. J. Math. 44, 102–121 (1922; JFM 48.1254.01)] considered a more general type of convergence based on directed sets, which was to be called nets, as coined by *J. L. Kelley* in [Duke Math. J. 17, 277–283 (1950; Zbl 0038.27003)]. *G. Birkhoff* [Bull. Am. Math. Soc. 41, 636 (1935; JFM 61.0641.20)] and *H. Cartan* [C. R. Acad. Sci., Paris 205, 595–598 (1937; Zbl 0017.24305); ibid. 205, 777–779 (1937; Zbl 0018.00302)] introduced the notion of filter convergence. As is well known, the idea of filter convergence was central in [Zbl 0027.14301]. The history of the axiomatization of convergence in topology culminated in the Manes-Barr characterization of a topological space in terms of an abstract ultrafilter convergence relation in [Zbl 0186.02901; Zbl 0289.54003; Zbl 0204.33202]. This is one of the two main streams leading to this book.

The other stream leading to this book emerged out from the epoch-making paper ["Metric spaces, generalized logic, and closed categories", Rend. Semin. Mat. Fis. Milano 43, 135–166 (1974; Zbl 0335.18006)], where *F. W. Lawvere* described metric spaces as categories enriched over the extended non-negative half-line adorned with his unique characterization of Cauchy completeness. The two streams were generalized by a monad $\mathbb{T}$ replacing the ultrafilter monad and a quantale or, more generally, a monoidal closed category $\mathcal{V}$ replacing the half-line. It was again Lawvere in 2000 who was the first to proposet that approach spaces in [Zbl 0891.54001] are to be described in terms of $\mathcal{V}$-multicategories in place of $\mathcal{V}$-categories, suggesting a merger of $\mathbb{T}$ and $\mathcal{V}$. Later, *M. M. Clementino* and *D. Hofmann* [Appl. Categ. Struct. 11, No. 3, 267–286 (2003; Zbl 1024.18003)], following a suggestion by Janelidze, gave a lax-algebraic description of approach spaces by using a numerical extension of the ultrafilter monad. Finally,*M. M. Clementino* and *W. Tholen* [J. Pure Appl. Algebra 179, No. 1–2, 13–47 (2003; Zbl 1015.18004)] succeeded in combining the two parameters efficiently.

The book under review consists of five chapters. The first chapter by *R. Lowen* and *W. Tholen* is an introduction, and the second chapter, "Monoidal structures" by *G. J. Seal* and *W. Tholen*, is a succinct introduction to category theory as well as the theory of ordered sets. The core of the book consists surely of the last three chapters.

Chapter III, "Lax algebras" by *D. Hofmann*, *G. J. Seal* and *W. Tholen*, deals with lax-algebraic description of topology, introduces the central category $(\mathbb{T},\mathcal{V})-\mathrm{Cat}$ whose objects are called $(\mathbb{T},\mathcal{V})$-algebras or $(\mathbb{T},\mathcal{V})$-spaces. As $(\mathbb{T},\mathcal{V})-\mathrm{Cat}$ fails to be Cartesian closed, the fourth section introduces its quasitopos extension $(\mathbb{T},\mathcal{V})-\mathrm{Grh}$ whose objects are called quasitopological spaces, in order to redeem exponentiability. The idea of quasitopological space is to be traced back to [Zbl 0031.28101]. The final section of the chapter is inspired by the equivalence between ordered compact Hausdorff spaces in [Zbl 0035.35402] and stably compact spaces in [Zbl 0452.06001] as well as the similar correspondence in the context of multicategories in [Zbl 0960.18004]. The exposition is largely influenced by [Zbl 1171.54025; Zbl 1173.18001].

Chapter IV, "Kleisli monoids" by *D. Hofmann*, *R. Lowen*, *R. Lucyshyn-Wright* and *G. J. Seal*, embarks upon another description of $(\mathbb{T},\mathcal{V})-\mathrm{Cat}$ as the category $\mathbb{T}-\mathrm{Mon}$ of monoids in the hom-set of a Kleisli category. After the well-known isomorphism

$$\mathbb{F}-\mathrm{Mon} \cong \mathrm{Top}$$

for the filter monad $\mathbb{F}$, the first section gives an isomorphism

$$\mathbb{T}\text{−Mon} \cong (\mathbb{T}, \mathbf{2})\ \text{−Cat}$$

The second section investigates sufficient conditions for a monad morphism $\alpha : \mathbb{S} \to \mathbb{T}$ to give rise to an isomorphism $(\mathbb{T}, \mathcal{V})\ \text{−Cat} \to (\mathbb{S}, \mathcal{V})\ \text{−Cat}$, as long as $\mathbb{S}$ and $\mathbb{T}$ are endowed with adequate lax extensions. Merging $\mathbb{T}$ and $\mathcal{V}$ into one entity, the third section introduces a new monad $\mathfrak{k} = \mathfrak{k}(\mathbb{T}, \mathcal{V})$, for which we have

$$(\mathbb{T}, \mathcal{V})\ \text{−Cat} \cong (\Pi, \mathbf{2})\ \text{−Cat}.$$

The fourth section identifies injective $(\mathbb{T}, \mathbf{2})$-categories with $\mathbb{T}$-algebras by recalling the fact that the forgetful functor $\text{Set}^{\mathbb{T}} \to (\mathbb{T}, \mathbf{2})\ \text{−Cat}$ is monadic of Kock-Zöberlein type. The fifth section is based largely upon [Zbl 1239.06003].

The first two sections of Chapter V, "Lax algebras as spaces" by *M. M. Clementino*, *E. Colebunders* and *W. Tholen*, explore such topological properties as separation, regularity, normality, extremal disconnectedness and compactness in the context of $(\mathbb{T}, \mathcal{V})$-categories as topological spaces. Emphasis is put on properties arising naturally in the $(\mathbb{T}, \mathcal{V})$-setting such as the symmetric descriptions of Hausdorff separation and compactness. The power of $(\mathbb{T}, \mathcal{V})$-spaces stems from their equational description as Eilenberg-Moore algebras. The central role of proper and open maps is highlightened in the third section, where proper and open maps in $(\mathbb{T}, \mathcal{V})\ \text{−Cat}$ are also considered equationally. Closure of properties under direct products, such as the Tychonoff theorem [JFM 55.0963.01] and its generalization called the Kuratowski-Mrówka theorem [Zbl 0003.10504; Zbl 0093.36305], is one of the outstanding themes in this section. The fourth section is concerned with an axiomatic approach to objects as spaces in a category provided with a class of proper maps. The last section investigates the notion of connectedness in $(\mathbb{T}, \mathcal{V})\ \text{−Cat}$.

All in all, the book is well written, and it will remain the standard textbook on the area for a long time.

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MSC:

18-00	Reference works (category theory)
18-02	Research monographs (category theory)
18B30	Categories of topological spaces and continuous mappings
18C20	Algebras and Kleisli categories associated with monads
18D10	Monoidal, symmetric monoidal and braided categories
54B30	Categorical methods in general topology

Keywords:

lax algebra; category theory; topology; quantale; monad; Eilenberg-Moore category; monoid; proper map; open map; compactness; Hausdorff separation; Kleisli category; monoidal closed category

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