**Gálvez-Carrillo, Imma**; **Kock, Joachim**; **Tonks, Andrew**

**Decomposition spaces, incidence algebras and Möbius inversion. II: Completeness, length filtration, and finiteness.** (English) Zbl 1403.18016

Adv. Math. 333, 1242-1292 (2018).

In the first part of this trilogy [Adv. Math. 331, 952–1015 (2018; Zbl 1403.00023)] the authors have introduced the notion of decomposition space as a generalization for incidence coalgebras, which is equivalent to the notion of unital 2-Segal space of [*T. Dyckerhoff* and *M. Kapranov*, "Higher Segal spaces. I", Preprint, arXiv:1212.3563]. The principal objective in this second part is to establish a Möbius inversion principle within the framework of *complete* decomposition spaces, and to analyze their associated finiteness conditions to ensure incidence coalgebras and Möbius inversion descent to classical level of $\mathbb{Q}$-vector spaces on taking the homotopy cardinality of the objects involved, which results in *Möbius decomposition spaces* as a generalization of Möbius categories in [*P. Leroux*, Cah. Topologie Géom. Différ. Catégoriques 16, 280–282 (1976; Zbl 0364.18001)]. Möbius decomposition spaces cover more coalgebra constructions than Möbius categories, comprehending the Faà di Bruno and Connes-Kremer bialgebras.

Reviewer: Hirokazu Nishimura (Tsukuba)

**MSC:**

| | |
|---|---|
| 18G30 | Simplicial sets; simplicial objects in a category |
| 16T10 | Bialgebras |
| 06A11 | Algebraic aspects of posets |
| 05A19 | Combinatorial identities, bijective combinatorics |
| 55U35 | Abstract homotopy theory; axiomatic homotopy theory |



**Keywords:**

decomposition space; 2-Segal space; incidence algebra; Möbius inversion; homotopy cardinality; length filtration

**Full Text:** DOI

**References:**

[1] Baez, J. C.; Dolan, J., From finite sets to Feynman diagrams, (Mathematics Unlimited—2001 and Beyond, (2001), Springer Berlin), 29-50 · Zbl 1004.18001

[2] Baez, J. C.; Hoffnung, A. E.; Walker, C. D., Higher dimensional algebra VII: groupoidification, Theory Appl. Categ., 24, 489-553, (2010) · Zbl 1229.18003

[3] Berger, C.; Melliès, P.-A.; Weber, M., Monads with arities and their associated theories, J. Pure Appl. Algebra, 216, 2029-2048, (2012) · Zbl 1256.18004

[4] Content, M.; Lemay, F.; Leroux, P., Catégories de Möbius et fonctorialités: un cadre général pour l'inversion de Möbius, J. Combin. Theory Ser. A, 28, 169-190, (1980) · Zbl 0449.05004

[5] Dür, A., Möbius functions, incidence algebras and power series representations, Lecture Notes in Mathematics, vol. 1202, (1986), Springer-Verlag Berlin · Zbl 0592.05006

[6] Dyckerhoff, T.; Kapranov, M., Higher Segal spaces I, in: Springer Lecture Notes in Mathematics, to appear

[7] Fiore, T. M.; Lück, W.; Sauer, R., Finiteness obstructions and Euler characteristics of categories, Adv. Math., 226, 2371-2469, (2011) · Zbl 1242.18014

[8] Gálvez-Carrillo, I.; Kock, J.; Tonks, A., Groupoids and faà di bruno formulae for Green functions in bialgebras of trees, Adv. Math., 254, 79-117, (2014) · Zbl 1295.16022

[9] Gálvez-Carrillo, I.; Kock, J.; Tonks, A., Decomposition spaces, incidence algebras and Möbius inversion, (Old omnibus version, not intended for publication). Preprint · Zbl 1403.00023

[10] Gálvez-Carrillo, I.; Kock, J.; Tonks, A., Homotopy linear algebra, Proc. Roy. Soc. Edinburgh Sect. A, 148, 293-325, (2018) · Zbl 06854604

[11] Gálvez-Carrillo, I.; Kock, J.; Tonks, A., Decomposition spaces, incidence algebras and Möbius inversion I: basic theory,

Adv. Math., 331, 952-1015, (2018) · Zbl 1403.00023

[12] Gálvez-Carrillo, I.; Kock, J.; Tonks, A., Decomposition spaces, incidence algebras and Möbius inversion III: the decomposition space of Möbius intervals, Adv. Math., (2018), in press · Zbl 1403.00023

[13] Gálvez-Carrillo, I.; Kock, J.; Tonks, A., Decomposition spaces and restriction species, Preprint

[14] Gálvez-Carrillo, I.; Kock, J.; Tonks, A., Decomposition spaces in combinatorics, Preprint

[15] Haigh, J., On the Möbius algebra and the Grothendieck ring of a finite category, J. Lond. Math. Soc. (2), 21, 81-92, (1980) · Zbl 0417.18005

[16] Joni, S. A.; Rota, G.-C., Coalgebras and bialgebras in combinatorics, Stud. Appl. Math., 61, 93-139, (1979) · Zbl 0471.05020

[17] Joyal, A., Foncteurs analytiques et espèces de structures, (Combinatoire énumérative, Montréal/Québec, 1985, Lecture Notes in Mathematics, vol. 1234, (1986), Springer Berlin), 126-159

[18] Joyal, A., Quasi-categories and kan complexes, J. Pure Appl. Algebra, 175, 207-222, (2002) · Zbl 1015.18008

[19] Joyal, A., The theory of quasi-categories, (Advanced Course on Simplicial Methods in Higher Categories, vol. II, Quaderns, vol. 45, (2008), CRM Barcelona), available at:

[20] Kock, J., Categorification of Hopf algebras of rooted trees, Cent. Eur. J. Math., 11, 401-422, (2013) · Zbl 1270.16030

[21] Kock, J., Perturbative renormalisation for not-quite-connected bialgebras, Lett. Math. Phys., 105, 1413-1425, (2015) · Zbl 1325.81126

[22] Kock, J., Polynomial functors and combinatorial Dyson-Schwinger equations, J. Math. Phys., 58, (2017), 36 pp. · Zbl 06712317

[23] Lawvere, F. W.; Menni, M., The Hopf algebra of Möbius intervals, Theory Appl. Categ., 24, 221-265, (2010) · Zbl 1236.18001

[24] Leinster, T., Notions of Möbius inversion, Bull. Belg. Math. Soc., 19, 911-935, (2012) · Zbl 1269.18002

[25] Leroux, P., LES catégories de Möbius, Cah. Topol. Géom. Différ., 16, 280-282, (1976) · Zbl 0364.18001

[26] Leroux, P., The isomorphism problem for incidence algebras of Möbius categories, Illinois J. Math., 26, 52-61, (1982) · Zbl 0487.18008

[27] Lück, W., Transformation groups and algebraic \textit{K}-theory, Lecture Notes in Mathematics, vol. 1408, (1989), Springer-Verlag Berlin · Zbl 0679.57022

[28] Lurie, J., Higher topos theory, Annals of Mathematics Studies, vol. 170, (2009), Princeton University Press Princeton, NJ, available from: · Zbl 1175.18001

[29] Lurie, J., Higher algebra, (2013), Available from

[30] Rota, G.-C., On the foundations of combinatorial theory. I. theory of Möbius functions, Z. Wahrsch. Verw. Gebiete, 2, 340-368, (1964) · Zbl 0121.02406

[31] Schmitt, W. R., Hopf algebras of combinatorial structures, Canad. J. Math., 45, 412-428, (1993) · Zbl 0781.16026

[32] Stanley, R. P., Enumerative combinatorics. vol. I, (1986), Wadsworth \& Brooks/Cole Advanced Books \& Software Monterey, CA, with a foreword by Gian-Carlo Rota · Zbl 0608.05001

[33] Stern, M., Semimodular lattices: theory and applications, Encyclopedia of Mathematics and Its Applications, vol. 73, (1999), Cambridge University Press Cambridge · Zbl 0957.06008

[34] Weber, M., Generic morphisms, parametric representations and weakly Cartesian monads, Theory Appl. Categ., 13, 191-234, (2004) · Zbl 1062.18008

[35] Weber, M., Familial 2-functors and parametric right adjoints, Theory Appl. Categ., 18, 665-732, (2007) · Zbl 1152.18005