Quillen, Daniel Gray

Formal theory of linear overdetermined systems of partial differential equations. (English)
Zbl 1295.35005
Cambridge, MA: Harvard University (Diss.). iii, 95 p. (1964).

This dissertation was written on overdetermined systems of linear partial differential equations under the direction of Raoul Bott. It was influenced very much by private communications with Shlomo Sternberg, D. C. Spencer and David Mumford. The dissertation is one of the earliest papers that applied the so-called Spencer cohomology to differential equations.

The cohomological approach to differential equations was pioneered by *D. C. Spencer* [Bull. Am. Math. Soc. 75, 179–239 (1969; Zbl 0185.33801)] as well as by his collaborators *H. Goldschmidt* [Ann. Math. (2) 86, 246–270 (1967; Zbl 0154.35103); J. Differ. Geom. 1, 269–307 (1967; Zbl 0159.14101)] and Daniel Gray Quillen (this dissertation), though the so-called Spencer cohomology had already been exploited in his deformation theory [*D. C. Spencer*, Ann. Math. (2) 76, 306–398, 399–445 (1962; Zbl 0124.38601)], to which this dissertation refers. It is to be noted that some of the ideas of the Spencer cohomology had already appeared in [*H. H. Johnson*, Proc. Am. Math. Soc. 16, 123–125 (1965; Zbl 0142.28203)] and [*H. H. Johnson*, Pac. J. Math. 17, 423–434 (1966; Zbl 0173.11702)]. It was in [*I. M. Singer* and *S. Sternberg*, J. Anal. Math. 15, 1–114 (1965; Zbl 0277.58008)] that the duality between the Spencer cohomology and the Koszul homology was first noted. It is interesting to remark that the authors Singer and Sternberg attributed it to private discussions with Alexander Grothendieck and Mumford.

It is in the Cartan-Kähler theory of exterior differential systems that the Cartan test has its origin. Its classical proof can be found in [*M. Janet*, Ann. Sci. Éc. Norm. Supér. (3) 41, 27–65 (1924; JFM 50.0321.03)]. The Cartan test in the modern homological guise is due to [*Y. Matsushima*, Nagoya Math. J. 6, 1–16 (1953; Zbl 0052.31905)] and [*Y. Matsushima*, in: Colloque de Topologie de Strasbourg, Années 1954–1955, 17 p. (1955; Zbl 0068.02801)]. It was in a letter of Serre appended to [*V. W. Guillemin* and *S. Sternberg*, Bull. Am. Math. Soc. 70, 16–47 (1964; Zbl 0121.38801)] that the infinite form of the dual Cartan test was established. Quillen provided its effective version in the appendix of the dissertation, and a bit different proof was provided in [*B. Malgrange*, Contemp. Math. 331, 193–205 (2003; Zbl 1062.58004)].

We close this review by letting Quillen speak out.

“In Chapter II we construct from a differential equation a sequence of first order differential operators called the k-th naive sequence.”

“The difficulty with the naive sequence is that although the original equation can be elliptic, the naive sequence is almost never elliptic in the sense that the sequence at every non-zero cotangent vector is exact. To remedy this difficulty, Spencer [Zbl 0185.33801] (the reviewer: Quillen has presumably made an error by referring to [*L. Hörmander*, Math. Ann. 140, 169–173 (1960; Zbl 0093.28903)]) developed a different method for constructing a resolution of the sheaf of solutions of an equation, which we study in Chapter III. §12 contains Bott’s construction of the Spencer sequence, and in §13 we show that its characteristics are the same as the original equation. In §14 we show that under certain hypotheses the Spencer sequence is formally exact, and that it has the same cohomology as the k-th naive sequence for k large.”

“In Chapter IV we use the Cartan-Kähler method to prove that for analytic equations the Spencer sequence and the stable naive sequences are exact, provided we restrict our attention to analytic equations. It is possible to prove the results of this chapter directly from the Cartan-Kähler theorem, however, we have chosen to reprove this theorem in the linear case by formulating the initial value problem, showing it has a unique solution, and then applying the Cauchy-Kowalewski theorem to prove this formal solution converges.”

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

35-02 Research monographs (partial differential equations)
35Nxx Overdetermined systems of PDE
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Keywords:

Spencer cohomology; Cartan-Kähler theory; Koszul homology