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Supergeometry, super Riemann surfaces and the superconformal action functional. (English) Zbl 1423.81006

Lecture Notes in Mathematics 2230. Cham: Springer (ISBN 978-3-030-13757-1/pbk; 978-3-030-13758-8/ebook). xiii, 303 p. (2019).

This monograph is concerned with the question how the superconformal action functional, a supersymmetric extension of the harmonic action functional on Riemann surfaces, is related to super Riemann surfaces and their moduli, laying the groundwork of the moduli space of super Riemann surfaces via the superconformal action functional and showing fantastic similarities to the theory of Riemann surfaces and harmonic maps.

The superconformal action functional appeared in the context of string theory already in the 1970s [*S. Deser* and *B. Zumino*, "A complete action for the spinning string" Phys. Lett. B 65, No. 4, 369–373 (1976; doi:10.1016/0370-2693(76)90245-8)], when the mathematical theory of supergeometry developed. Super Riemann surfaces, a supergeometric analogue of Riemann surfaces, appeared only a little later, their significance for string theory being appreciated [*D. Friedan*, in: Unified string theories, Workshop, Santa Barbara/Calif. 1985, 162–213 (1986; Zbl 0648.53057); *E. D'Hoker* and *D. H. Phong*, "The geometry of string perturbation theory", Rev. Modern Phys. 60, No. 4, 917–1065 (1988; doi:10.1103/revmodphys.60.917)].

The principal objective in this monograph is to build a bridge between the mathematical theory of supergeometry and super Riemann surfaces with the superconformal action functional motivated by physics. The reader is strongly referred to *E. Witten*'s [Pure Appl. Math. Q. 15, No. 1, 517–607 (2019; Zbl 1421.81102); ibid. 15, No. 1, 213–516 (2019; Zbl 1421.81101); ibid. 15, No. 1, 57–211 (2019; Zbl 1423.32012); ibid. 15, No. 1, 3–56 (2019; Zbl 1421.58001)].

The monograph consists of 13 chapters and two appendices.

Chapter 2 is concerned with linear superalgebra. There are, roughly speaking, three approaches to supermanifolds, namely, the Rogers-DeWitt approach, the approach via ringed spaces (also called the Berezin-Kostant-Leites approach) and the approach via functors of points. This monograph adopts the ringed space approach, because the ultimate goal of the book lies in moduli spaces of super Riemann surfaces.

Chapter 3 gives the category of supermanifolds from this standpoint, discussing families of supermanifolds and base change in the spirit of [*P. Deligne* and *J. W. Morgan*, in: Quantum fields and strings: a course for mathematicians. Vols. 1, 2. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. Providence, RI: AMS, American Mathematical Society. 41–97 (1999; Zbl 1170.58302)]. Underlying even manifold is also discussed with its existence theorem (Theorem 3.3.7) and the statement (Corollary 3.3.14) that a superdiffeomorphism induces a diffeomorphism on the underlying even manifold and a change of the embedding.

Chapter 4 explains the generalization of vector bundles to families of supermanifolds.

Chapter 5 gives an introduction to the theory of super Lie groups.

Chapter 6 addresses the theory of principal bundles and connections on them for families of supermanifolds, following quite closely what is already standard in classical differential geometry.

Chapter 7 introduces the theory of smooth families of complex supermanifolds. Families of complex supermanifolds are locally given by $\mathbb{C}^{m|n}$ and patched by smooth families of holomorphic coordinate changes. A super version of the Newlander-Nirenberg theorem [*A. McHugh*, J. Math. Phys. 30, No. 5, 1039–1042 (1989; Zbl 0679.58008); *A. Yu. Vaĭntrob*, Vopr. Teor. Grupp Gomologicheskoj Algebry 1985, 139–142 (1985; Zbl 0735.53025)] applies to families of supermanifolds.

Chapter 8 is concerned with the theory of integration for families $M \to B$ of supermanifolds.

Chapter 9 describes super Riemann surfaces and the additional structure of an adapted metric with the help of reductions of the structure group.

Chapter 10 studies the torsion tensor of connections on the reduction of the frame bundle of a super Riemann surface M to $\mathrm{Tr}_{\mathbb{C}}(1 \mid 1)$, SCL and U(1).

Given an embedding $i : |M| \to M$ of the underlying even manifold $|M|$ into a super Riemann surface M, Chapter 11 deals with the structure induced on $|M|$. It is shown that a given -structure on induces a metric g, a spinor bundle S and a differential form χ with values in S, called gravitino, on $|M|$.

Chapter 12 is concerned with the superconformal action functional. It is shown that the action functional $A(\Phi, m)$ with a map $\Phi : M \to N$ from a super Riemann surface M to an arbitrary Riemannian submanifold N and a U(1)-metric m on M is a natural generalization of the action functional of harmonic maps on Riemann surfaces in several respects.

Chapter 13 collects different calculations in the Wess-Zumino gauge.

Appendix A gathers some formulas and well-known results on Clifford algebras, spinors and spin structures in 2-dimension with fixing notation and sign conventions. Appendix B aims to give a direct verification of the invariance of the action functional $A(\varphi, g, \psi, \chi, F)$ under the supersymmetry transformations.

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MSC:

81-02	Research monographs (quantum theory)
81T30	String and superstring theories
53Z05	Applications of differential geometry to physics
14D21	Applications of vector bundles and moduli spaces in mathematical physics
81T60	Supersymmetric field theories
14M30	Supervarieties
17A70	Superalgebras
58A50	Supermanifolds, etc. (global analysis)

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