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Monoidal categories and topological field theory. (English) Zbl 1423.18001

Progress in Mathematics 322. Basel: Birkhäuser/Springer (ISBN 978-3-319-49833-1/hbk; 978-3-319-49834-8/ebook). xii, 523 p. (2017).

The study of monoidal categories was initiated in [*S. MacLane*, Rice Univ. Stud. 49, No. 4, 28–46 (1963; Zbl 0244.18008)] and [*J. Benabou*, C. R. Acad. Sci., Paris 256, 1887–1890 (1963; Zbl 0111.02201)]. The intimate relationship between monoidal categories and topology was sparked by *R. Penrose* [Proc. Conf. Math. Inst. 1969, 221–244 (1971; Zbl 0216.43502)]. Morphisms in monoidal categories and operations on them may be represented by pictures, which leads to a fantastic graphical calculus allowing to replace lengthy algebraic computations by simple topological arguments. Monoidal categories play a significant role in quantum topology, an arena of mathematics and theoretical physics founded by Vaughan Jones and Edward Witten in the 1980s, which studies Topological Quantum Field Theory (TQFT) as well as related invariants such as knots, links manifolds, etc. This monograph is engaged in connections between monoidal categories and 3-dimensional TQFTs.

The first named author of the monograph introduced two fundamental constructions of 3-dimensional TQFTs in collaboration with Nikolai Reshetikhin and Oleg Viro in the late 1980s, the Reshetikhin-Turaev construction [*N. Reshetikhin* and *V. G. Turaev*, Invent. Math. 103, No. 3, 547–597 (1991; Zbl 0725.57007)] making use of surgery on 3-manifolds to derive a TQFT from a modular category while the Turaev-Viro construction [*V. G. Turaev* and *O. Y. Viro*, Topology 31, No. 4, 865–902 (1992; Zbl 0779.57009)] exploiting state sums on on skeletons of 3-manifolds to derive a TQFT from a spherical fusion category. The latter construction originally involved representations of quantum groups, whose categorical formulation was given in [*J. W. Barrett* and *B. W. Westbury*, Trans. Am. Math. Soc. 348, No. 10, 3997–4022 (1996; Zbl 0865.57013)]. In 1995 the first named author of the monograph conjectured that the state sum TQFT derived from a spherical fusion category is isomorphic to the surgery TQFT derived from the Drinfeld-Joyal-Street center of that category. The principal objective in this monograph is to establish this conjecture.

The monograph is divided into four parts. Part I is an introduction to monoidal categories, consisting of five chapters (Chapters 1–5). Chapter 1 reviews the basics of monoidal categories. Chapter 2 addresses the method of representing morphisms in categories by planar diagrams. Chapter 3 reviews braidings and symmetris in monoidal categories, discussing the graphical calculus for braided pivotal categories and defining ribbon categories. Chapter 4 introduces the classes of linear, pre-fusion, fusion and modular categories. Chapter 5 recalls the Drinfeld-Joyal-Street center of a monoidal category. Part I culminates in two fundamental theorems (Theorems 5.3 and 5.4) of *M. Müger* on the centers of additive pivotal fusion categories [J. Pure Appl. Algebra 180, No. 1–2, 159–219 (2003; Zbl 1033.18003), Theorems 1.2 and 3.16] being only stated with their proofs deferred to Chapter 9.

Part II is concerned with Hopf algebras and monads, consisting of 4 chapters (Chapters 6–9). Chapter 6 investigates Hopf algebras in braided categories, paving the way to Hopf monads [*A. Bruguières* et al., Adv. Math. 227, No. 2, 745–800 (2011; Zbl 1233.18002); *A. Bruguières* and *A. Virelizier*, Adv. Math. 215, No. 2, 679–733 (2007; Zbl 1168.18002)] in Chapter 8. In Chapter 7, the basics of the theory of monads and bimonads is outlined. Chapter 9, based on [*A. Bruguiéres* and *A. Virelizier*, Trans. Am. Math. Soc. 364, No. 3, 1225–1279 (2012; Zbl 1288.18004); Pac. J. Math. 264, No. 1, 1–30 (2013; Zbl 1282.18005)], shows that, under mild hypothesis, the center of a rigid category is the category of modules over a quasitriangular Hopf monad, allowing of computation of the coend of the center of a pivotal fusion category.

Part III, consisting of 4 chapters (Chapters 10–13), gives a formal definition of a TQFT and constructs the state sum TQFT associated with a spherical fusion category. Chapter 10 is concerned with fundamentals of TQFT. Chapter 11 defines skeletons of compact 3-manifolds and study their local transformations. Chapter 12 associates with each pivotal $\boldsymbol{k}$-category a family of modules (called multiplicity modules), introducing an invariant of colored graphs with values in tensor products of multiplicity modules, which is the main tool in the construction of a 3-dimensional TQFT in Chapter 13.

Part IV, consisting of 4 chapters (Chapters 14–17), extends the state sume TQFT to a so-called graph TQFT, which applies to 3-manifolds with colored ribbon graphs. Chapter 14 defines ribbon graphs in terms of so-called plexuses, explaining how to present ribbon graphs by diagrams on skeletons of the manifold and introducing local moves on the diagram with the associated ribbon graphs preserved. The main results in this chapter (Theorems 14.3 and 14.4) claim that any diagram of isotopic ribbon graphs may be related by these local moves. Chapter 14 derives from a spherical fusion category with invertible dimension a graph TQFT applicable to colored ribbon graphs in 3-dimensional cobordisms, cobordisms with empty graphs degenerating into the TQFT of the previous chapter. Chapter 16 establishes several properties of the state sum graph TQFT, two of which (Theorems 16.1 and 16.2) are used in the succeeding chapter. Theorem 17.1 of Chapter 17 is the main theorem of Part IV, interpreting the state sum graph TQFT as a surgery TQFT.

9 appendices are accompanied:

(A) Examples of monoidal categories
(B) Coends
(C) Abelian categories
(D) Hopf monads vs Hopf algebras
(E) Unordered tensor products of modules
(F) The $6j$ symbols
(G) Unitary TQFTs
(H) The Dijkgraaf-Witten invariants
(I) Hnits and solutions to exercises

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18-02 Research monographs (category theory)
18D10 Monoidal, symmetric monoidal and braided categories
16T05 Hopf algebras and their applications
81T45 Topological field theories

Full Text: DOI