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Two problems in the theory of differential equations. (English. Russian original) Zbl 07097227

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This paper gives examples where supersymmetry and nonholonomicity are manifested in very different, somewhat unexpected situations. The paper consists of three sections.

The first section is concerned historically with Faddeev-Popov ghosts as analogues of momenta in *odd* mechanics. The author's papers [*D. A. Leites*, Sov. Math., Dokl. 18, 1277–1280 (1977; Zbl 0403.17002); translation from Dokl. Akad. Nauk SSSR 236, 804–807 (1977)]; textitD. A. Leites and *V. V. Minachin*, in: The formalism of left and right connections on supermanifolds. Proc. Int. Workshop Quantum Field Theory, Varna/Bulg., 14–17 (1989; Zbl 0763.17002)] proclaimed the existence of an *odd* analogue of mechanics, but no physicists listened to him in the 1970s. However, the very *odd* mechanics was rediscovered in Batalin and Vilkovisky's [Phys. Lett. B, 102, 27–31 (1981); Phys. Rev. D, 28, 2567–2582 (1982); Erratum ibid. 30, 508 (1984)], which has attracted huge interest from physicists because of a plenty of applications demonstrated in the first paper. The Faddeev-Popov ghosts were interpreted, and mathematicians became interested in derivations of the Schouten construction called *Batalin-Vilkovisky algebras*.

The second section is concerned with supersymmetries of differential equations. As is well known, Sophus Lie introduced Lie groups to do for differential equations an analogue of what Galoid did for algebraic equations. From the standpoint of supersymmetry, É. Cartan took a decisive step by reformulating the notion "differential equations" in terms of exterior differential forms. However, Cartan did not know that the differential ideal distinguishes a certain subsuperscheme or subsupervariety in the affine superspace. The author proposes the following problem:

- Describe supersymmetry groups of differential equations of mathematical physics. Which of them do not reduce to groups?

The author also discusses the following topics:

- gauge fields from the super standpoint;
- super specifics ([*Yu. I. Manin*, Gauge fields and complex geometry (Russian). Moskva: "Nauka" (1984; Zbl 0576.53002); Gauge field theory and complex geometry. Transl. from the Russian by N. Koblitz and J. R. King. Berlin etc.: Springer-Verlag (1988; Zbl 0641.53001); 2nd ed. Berlin: Springer (1997; Zbl 0884.53002)], [*P. Deligne* (ed.) et al., Quantum fields and strings: a course for mathematicians. Vols. 1, 2. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. Providence, RI: AMS, American Mathematical Society (1999; Zbl 0984.00503)]);
- supersymmetry of Maxwell and Dirac equations;
- Witten discoveries [Phys. Lett. B, 77, 394–398 (1978); *E. Witten*, Nucl. Phys., B 188, No. 3, 513–554 (1981; Zbl 1258.81046)].

The third section is concerned with formal integrability of differential systems and nonholonomic structures. Criteria for the formal integrability of differential equations were expressed in [*R. L. Bryant* et al., Exterior differential systems. New York etc.: Springer-Verlag (1991; Zbl 0726.58002)] in terms of the Lie algebra $\mathfrak{g}$ of local symmetries of the differential and the certain Spencer cohomology of $\mathfrak{g}$, where $\mathfrak{g}$ is assumed to be a $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{g} = \bigoplus_{i \geq -d} \mathfrak{g}_i$$

of depth $d = 1$. The author proposes the following questions:

- Why does nobody among all those who solve differential equations every day ever use this approach in place of numerical approximations and scores of other methods?

- According to a well-known theorem, the symmetries of a given differential equation are induced by either point or contact transformations. Why do the integrability criteria of differential equations given in [Zbl 0726.58002] and later works deal with differential equations of the first type only?

To generalize the criteria for formal integrability to embrace the differential equations whose symmetries are induced by contact transformations, new methods and results are needed. First, we need the generalization of Cartan prolongs for Lie superalgberas of depth $d > 1$ due to [*I. M. Shchepochkina*, Theor. Math. Phys. 147, No. 3, 821–838 (2006; Zbl 1177.17015); translation from Teor. Mat. Fiz. 147, No. 3, 450–469 (2006)]. Second, we need the definition of an analogue of Spencer cohomology for Lie superalgeba of depth $d > 1$ [*D. Leites*, Homology Homotopy Appl. 4, No. 2(2), 397–407 (2002; Zbl 1015.53012)]. The author also discusses nonholonomic structures and their corresponding analogues of the curvature tensor.

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MSC:

58A15 Exterior differential systems (Cartan theory)
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