Fractional Linear Birth-Death Stochastic Process
- An application of Heun's differential equation

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Abstract

The method of Heun's differential equation is demonstrated in studying a fractional linear birth-death process (FLBDP) with long-memory described by a master equation. The exact analytic solution of the generating function for the probability density is obtained on the basis of Heun's differential equation. The multi-fractal nature of FLBDP associated with long-memory is demonstrated in conjunction with the present simple birth-death process. Finally, the subtle multi-fractal nature of *critical fluctuations* under long-memory is also displayed in the present FLBDP. Further, discussions are also given on the features of transient fluctuation in systems with long-memory.

Keywords: Fractional linear birth-death process, Master equation, Generating function, the waiting time (lifetime) distribution, critical fluctuations

1 Introduction

The exactly solvable Markovian birth-death stochastic processes have been studied and classified by Karlin and McGregor [1, 2] in the context of the orthogonal polynomials. Their theory has been applied extensively in the field of population biology (e.g., Goel and Dyn,[3]), in the theory of queues (e.g., Cox, [4]) and in intercellular transport (e.g., Schuss, Singher and Holman, [5] ; Bressloff, [6]). Even when a nonlinear interaction is explicitly accounted in the birth λ_n and the death μ_n rate, the Markovian master equation,

$$\frac{d}{dt}p(n,t) = \lambda_{n-1}\ p(n-1,t) + \mu_{n+1}\ p(n+1,t)$$

$$-(\lambda_n + \mu_n)\ p(n,t)\ , \quad (n \geq 1) \tag{1}$$

and

$$\frac{d}{dt}p(0,t) = \mu_1\ p(1,t) - (\lambda_0 + \mu_0)\ p(0,t)\ , \quad (n = 0) \tag{2}$$

can be solved exactly for a class of nonlinear functions (λ_n, μ_n) of the random variable n (cf., Ismail [7], Schoutens [8], Sasaki [9]).

Eq. (1) without nonlinear birth-death rates has been also extensively studied in quantum optics [10]; quantum chromodynamics (QCD) and Hadron interactions [11]; enter-exit stochastic dynamics of market traders to describe macroeconomics [12]; the stochastic transport problem for PSD-95 in biological cells [13]; the stochastic birth-death dynamics of solitons and holes [14]; ecology [15] and epidemiology [16], and the stochastic dynamics of neural spike generation [17].

We have expected that the effects of memory should be accounted for in studies on [a] optical rogue waves and related extreme-event statistics ([18] and [19]); [b] QCD statistics [20]; [c] economical time series data [21, 22, 23, 24]; [d] transport problems in cell biology [25, 26], [e] anomalous diffusion of solitons; [27] and [e] ecology and epidemiology [28].

In previous papers, the fractional Poisson process (FPP), [29, 30] the fractional generalized birth process (FGBP) [31, 32] and a linear birth-death process [33] were studied. It was shown that the generating function u

(GF) for the FGBP with $\lambda_n = \lambda n+\nu$ and $\mu_n = 0$ in the Laplace domain is described by the Gauss hypergeometric equation

$$x(1-x)\frac{d^2u}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{du}{dx} - \alpha\beta u = 0 \ , \tag{3}$$

where the concrete form of α, β and γ are given in Eq. (B3) in Appendix B of Ref. [32].

Also it was shown that the Laplace transformed generating function (GF) for a simple fractional birth-death process with $\lambda_n = \nu$ and $\mu_n = n\mu$ in Eq. (1) [34] reduces to the confluent Heun's equation [35, 36, 37]:

$$\frac{d^2u}{dx^2} + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}\right)\frac{du}{dx} + \frac{\alpha p x - q}{x(x-1)}u = 0 \ , \tag{4}$$

where the concrete form of the coefficients of $(\alpha, \beta, \gamma, p, q)$ are given in Eq. (33) of Ref. [34].

The aim of the present paper is first to show that the GF for Eq. (1) in the Laplace domain satisfies the Heun's equation (HE) in the canonical form: [35, 36]

$$\frac{d^2u}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\frac{du}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-a)}u = 0 \ , \tag{5}$$

where the concrete form of the coefficients ($\gamma, \delta, \epsilon, \alpha, \beta$ and q) will be shown in section 3.

The HE has four true singular points at $x = 0, 1, a$, and ∞, one more true singular point than those of the Gauss hypergeometric function. Then it is shown that the solution of the generating function is expressed by the Mittag–Leffler function with the coefficient of the Meixner polynomials. Hence, the various statistical quantities are obtained analytically.

The paper is organized as follows. Section 2 describes the fractional linear birth-death process (FLBDP) to be analyzed. The relevance of the FLBDP to problems in physics is remarked briefly. Section 3 describes that the generating function takes the form of the HE in the canonical form. The formal solution by factorization is also introduced. The problems associated with solving the HE are also remarked on. The integral representation of the HE is given. The integral representation of the critical condition is presented. The representation in terms of the Meixner polynomials is also given. Section 4 presents (i) the expressions of moments (ii) moments under the critical condition (iii) the lifetime distribution for $n_0 \neq 0$, and (iv) the waiting time distribution under the critical condition. Section 5 discusses the method of parameter estimation. Then, through applications to some physical systems the properties expected from the FLBDP are demonstrated. Section 6 is devoted to a summary and remarks.

2 Fractional Birth-Death Master Equation

2.1 Model

Here we neglect the nonlinear terms, and consider a simple memory effect on the basis of a fractional linear birth-death process (FLBDP) in the form:

$$\begin{aligned} {}_0D_t^\alpha p_\alpha(n,t) &= [\lambda(n-1)+\nu]\ p_\alpha(n-1,t) + \mu(n+1)\ p_\alpha(n+1,t) \\ &-[(\lambda+\mu)n+\nu]\ p_\alpha(n,t)\ , (n \geq 1) \end{aligned} \tag{6}$$

and

$${}_0D_t^\alpha p_\alpha(0,t) = \mu\ p_\alpha(1,t) - \nu\ p_\alpha(0,t)\ , (n=0) \tag{7}$$

where $0 < \alpha \leq 1$, and the parameters λ, μ and ν are assumed to be constants. ${}_0D_t^\alpha$ is the Caputo fractional derivative defined by

$${}_0D_t^\alpha f(t) \equiv \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-\tau)^\alpha} D_\tau f(\tau)\ d\tau \ , \tag{8}$$

where $D_t = \frac{d}{dt}$ is the ordinary derivative. The initial condition of Eq. (6) is set to

$$p_\alpha(n,0) = \delta_{n,n_0}. \tag{9}$$

The introduction of the power law memory by the fractional derivative (8) is a special choice for describing the dynamical features in complex birth-death processes. In spite of the limitation in applications, one can catch analytically the features of power-law decay in waiting-time (or lifetime) distribution and that of the correlation function. Also, the counting statistics with the fractional power-law is ubiquitous in FPP, FGBP and FLBDP with a critical condition, and in many real experiments. Since it is quite hard to get high accuracy of power law of them in direct numerical simulation of the master equation in Eq. (6), the fractional generalization might be useful.

The steady state solution of Eq. (6) for $\mu > \lambda$ with $\nu > 0$ is given by

$$p_\alpha(n,\infty) = \frac{\Gamma(n+\frac{\nu}{\lambda})}{\Gamma(\frac{\nu}{\lambda})n!}\left(1-\frac{\lambda}{\mu}\right)^{\nu/\lambda}\left(\frac{\lambda}{\mu}\right)^n . \tag{10}$$

2.2 Physical Relevance to FLBDP

The model in Eq. (6) under the initial value $n_0 = 0$ was studied by Suzuki and Biyajima [20]. They have argued for the model with the observed charged multiplicities of $p\bar{p}$, e^+e^- and e^+p collision. However, their argument is not satisfactory since the initial condition n_0 is not controllable in their experiment.

The master equation in Eq. (6) without memory, $\alpha = 1$, has been also studied in discussing epidemic evolution (i.e., SIRXY model) [16]. The argument becomes more realistic if one takes into account the effect of spatial diffusion, which may incorporate the memory term as $(t-t')^{-1/2}$.

The effect of memory has been discussed in conjunction with the financial time series data. The model might be closely related to Cox–Ingersoll–Ross model [8, 12] with a long-memory, which are equivalent to the FLBDP in Eq. (6).

Also, the model with $\nu = 0$ and $n_0 = 1$ is studied by Orsingher and Polito [31] from the viewpoint of pure mathematics. The case may be related to 3D scroll wave which will be discussed in section 5. Konno discussed the model with $\lambda_n = \nu$ and $\mu_n = \mu n$ in conjunction with an optical rogue wave problem [34].

3 Generating Function

3.1 Method of Heun's Equation

The generating function $g_\alpha(z,t)$ for the process in Eq. (6) becomes

$${}_0D_t^\alpha g_\alpha(z,t) = [\lambda z(z-1)+\mu(1-z)]\frac{\partial}{\partial z}g_\alpha(z,t) + \nu(z-1)g_\alpha(z,t) , \tag{11}$$

where $g_\alpha(z,t)$ is defined by $g_\alpha(z,t) = \sum_{n=0}^\infty z^n p_\alpha(n,t)$. From the initial condition in Eq. (9), one obtains $g_\alpha(z,0) = z^{n_0}$.

After taking Laplace transform, one obtains

$$(\lambda z-\mu)(z-1)\frac{\partial}{\partial z}g_\alpha[z,s] + \nu(z-1)g_\alpha[z,s] = s^\alpha g_\alpha[z,s] - s^{\alpha-1}z^{n_0} , \tag{12}$$

where $g_\alpha[z,s] \equiv \int_0^\infty e^{-st}g_\alpha(z,t)dt$. Taking derivative with respect to z, one obtains the second order differential equation,

$$\frac{d^2}{dz^2}g_\alpha[z,s] + \left(\frac{\ell}{z}+\frac{h}{z-1}+\frac{k}{z-a}\right)\frac{d}{dz}g_\alpha[z,s] + \frac{ABz-q}{z(z-1)(z-a)}g_\alpha[z,s] = 0 , \tag{13}$$

where the true singularities are located at $z = (0, 1, a(=\frac{\mu}{\lambda}), \infty)$, and

$$\ell = -n_0, h = 1-\frac{s^\alpha}{\lambda-\mu}, k = 1+\frac{\nu}{\lambda}+\frac{s^\alpha}{\lambda-\mu} , AB = \frac{\nu}{\lambda} \text{ and } q = -n_0\left(\frac{\nu}{\lambda}+\frac{s^\alpha}{\lambda}\right) . \tag{14}$$

Eq. (13) is classified as the Heun's differential equation [35] with the parameters (ℓ, h, k, A, B, q). It is known that the corresponding Riemann's scheme is represented by

$$\begin{pmatrix} z=0 & z=1 & z=a & z=\infty \\ 0 & 0 & 0 & A \\ 1-\ell & 1-h & 1-k & B \end{pmatrix} . \tag{15}$$

The accessory parameter q in Eq. (14) is involved in Eq. (13), which is irrelevant to the characteristic exponents in Eq. (15). When the initial condition $n_0 = 0$ is selected, the values of q and ℓ take null values, so that Eq. (13) reduces to the hypergeometric equation with the true singularities located at $z = (1, a, \infty)$. It seems as if the exact analytic solution cannot be obtained when $n_0 \neq 0$. However, this is not the case. As is shown in the next section, the exact solution of Eq. (13) can be given for any value of n_0.

When the critical condition is satisfied, $\lambda = \mu$, the scaled generating function $G_\alpha[z,s]$ ($g_\alpha[z,s] = \exp(-s^\alpha/\lambda/(z-1))(\frac{1}{1-z})^{1-n_0} G_\alpha[z,s]$) in the Laplace domain satisfies the confluent Heun equation [35, 36]

$$\frac{d^2}{dy^2} G_\alpha[z,s] + \left(\ell + \frac{h}{y} + \frac{k}{y-1}\right)\frac{d}{dy} G_\alpha[z,s] + \frac{ABz - q}{y(y-1)} G_\alpha[y,s] = 0 \ , \tag{16}$$

where $y = \frac{1}{1-z}$,

$$\ell = \frac{s^\alpha}{\lambda} \ , h = 2 - n_0 - \frac{\nu}{\lambda} \ , k = -n_0 \ , AB = \left(1 - n_0 - \frac{\nu}{\lambda}\right)\frac{s^\alpha}{\lambda} \text{ and } q = \left(1 - n_0 - \frac{\nu}{\lambda}\right)\left(n_0 + \frac{s^\alpha}{\lambda}\right) \ . \tag{17}$$

3.2 Integral Representation of Heun's Function

Although the generating function $g_\alpha[z,s]$ is expressed by the general Heun's function $GH(a,q;\ell,h,k,A,B;z)$ [35], it is necessary to get the integral representation of it. Otherwise, it is not easy to perform analytical/numerical estimations of statistical quantities associated with the FLBDP in Eq. (6).

One can get an integral representation of the generating function from Eq. (12) as

$$g_\alpha[z,s] = -s^{\alpha-1}\left(\frac{z-1}{\lambda z - \mu}\right)^{\frac{s^\alpha}{\lambda-\mu}} (\lambda z - \mu)^{-\frac{\nu}{\lambda}} \int_0^z d\omega \ \omega^{n_0} (\omega - 1)^{-\frac{s^\alpha}{\lambda-\mu}-1} (\lambda\omega - \mu)^{\frac{s^\alpha}{\lambda-\mu}-1+\frac{\nu}{\lambda}} \ . \tag{18}$$

Ronveaux studied the factorization of Heun's difference operator $\mathcal{H}$ [36]. The factorizable solutions of the Heun's equations are classified into the 9 types (cf. Table 1 by Ronveaux [36]). The above solution is the class VI solution with the parameter as described below:

$$\mathcal{H} = [LD + M][\overline{L}D + \overline{M}] \ , \tag{19}$$

where

$$L = z, \quad \overline{L} = (z-1)(z-a) \ , \quad M = \ell, \quad \overline{M} = (h+k-2)z + 1 - k + a(1-h) \ ,$$
$$A = 1 + \ell, \quad B = h + k - 2, \quad q = -\ell(1 + a - ah - k) \ .$$

With the transformation of the state variable by

$$\frac{\omega}{z}\frac{z-1}{\omega-1} = 1 - x \ , \tag{20}$$

the generating function in Eq. (18) in the Laplace domain becomes (cf. Appendix A)

$$g_\alpha[z,s] = s^{\alpha-1}\frac{z^{n_0+1}}{\lambda z - \mu}\int_0^1 dx \ (1-x)^{n_0}(1-xz)^{-n_0-\frac{\nu}{\lambda}} \times \left[1 - \frac{(\lambda-\mu)xz}{\lambda z - \mu}\right]^{\frac{\nu}{\lambda}+\frac{s^\alpha}{\lambda-\mu}-1} . \tag{21}$$

As far as we know, this is the first time to obtain the explicit integral representation of the general Heun's function.

When the critical condition with $\lambda = \mu$ and $\nu \neq 0$ is satisfied under the initial condition n_0, $g_\alpha[z,s]$ in the Laplace domain (18) [or (21)] reduces to $g_\alpha(z,t)$ in the time domain as (cf. Appendix B)

$$g_\alpha(z,t) = \sum_{n=0}^{n_0} \binom{n_0}{n} (z-1)^n \int_0^\infty \exp(-\omega) E_{\alpha,1}^{n+\frac{\nu}{\lambda}}(-\omega\lambda(1-z)t^\alpha) \ d\omega \ , \tag{22}$$

where $E_{\alpha,\beta}^{\gamma}(x)$ is a generalized Mittag–Leffler function [42] defined by

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma)\Gamma(\alpha n + \beta)} x^n \ . \tag{23}$$

When a critical condition with $\lambda = \mu$ and $\nu = 0$ under the initial condition $n_0 = 1$ is considered, $g_\alpha(z,t)$ in the time domain in Eq. (22) reduces to

$$g_\alpha(z,t) = 1 + (z-1)\int_0^\infty \exp(-\omega) E_\alpha(-\omega\lambda(1-z)t^\alpha) \ d\omega \ . \tag{24}$$

The expression will be used to discuss the features of the fluctuations near the critical point for 3D scroll waves due to Clayton's numerical experiment [39, 40].

3.3 Representation by Meixner polynomials

The generating function in the time domain in Eq. (21) is rewritten in terms of the Mittag–Leffler function as

$$g_\alpha(z,t) = \left(\frac{\lambda z - \mu}{\lambda - \mu}\right)^{-\frac{\nu}{\lambda}} \sum_{n=0}^{\infty} \frac{M_n(n_0, \frac{\nu}{\lambda}, \frac{\lambda}{\mu})}{n!} \left(\frac{\lambda(z-1)}{\lambda z - \mu}\right)^n E_\alpha\big(n(\lambda-\mu)t^\alpha\big) \ , \tag{25}$$

where the $M_n(x;\beta,c)$ are the Meixner polynomials of the first kind,

$$M_n(x;\beta,c) = (\beta)_n F\Bigg([-x,-n],[\beta];1-\frac{1}{c}\Bigg), \tag{26}$$

where $F([\alpha,\beta],[\gamma];x)$ is the hypergeometric function, and $(\beta)_n = \frac{\Gamma(\beta+n)}{\Gamma(\beta)}$. Lengthy calculation is required to transform Eq. (18) in the Laplace domain to Eq. (25) in the time domain. So the simplest derivation of Eq. (25) with the use of the Lévy transform is given in Appendix C. The functional forms of the Meixner polynomials are

$$M_0(x;\beta,c) = 1, \quad M_1(x;\beta,c) = \beta + x\Bigg(1-\frac{1}{c}\Bigg),$$

$$M_2(x;\beta,c) = \beta(\beta+1) + \Bigg(1-\frac{1}{c}\Bigg)\Bigg(2\beta+1+\frac{1}{c}\Bigg)x + \Bigg(1-\frac{1}{c}\Bigg)^2 x^2 \ , \tag{27}$$

$$\cdots$$

$$M_n(x;\beta,c) = n!\sum_{k=0}^{n}\binom{x}{k}(-1)^k c^{-k}\binom{x+\beta+n-k-1}{n-k} \ . \tag{28}$$

4 Statistical Properties

4.1 Expressions of moments

Expressions for the moments can be obtained easily with the use of $g_\alpha[z,s]$ in Eq. (21): $\langle n[s]\rangle = g'_\alpha[z=1,s]$, $\langle n[s]^2\rangle = g''_\alpha[z=1,s] + g'_\alpha[z=1,s]$ and so on, and taking the inverse Laplace transform of them. It is more convenient to take derivative with respect to z in Eq. (25), and to express the moments in terms of the Meixner polynomials as

$$g'_\alpha(z=1,t) = \Bigg(\frac{\lambda}{\lambda-\mu}\Bigg)\Bigg[-\beta M_0\Bigg(n_0;\frac{\nu}{\lambda},\frac{\lambda}{\mu}\Bigg) + M_1\Bigg(n_0;\frac{\nu}{\lambda},\frac{\lambda}{\mu}\Bigg)E_\alpha((\lambda-\mu)t^\alpha)\Bigg]$$

and

$$g''_\alpha(z=1,t) = \Bigg(\frac{\lambda}{\lambda-\mu}\Bigg)^2\Bigg[(\beta+1)\beta M_0\Bigg(n_0;\frac{\nu}{\lambda},\frac{\lambda}{\mu}\Bigg) - 2(\beta+1)M_1\Bigg(n_0;\frac{\nu}{\lambda},\frac{\lambda}{\mu}\Bigg)E_\alpha((\lambda-\mu)t^\alpha) + M_2\Bigg(n_0;\frac{\nu}{\lambda},\frac{\lambda}{\mu}\Bigg)E_\alpha(2(\lambda-\mu)t^\alpha)\Bigg].$$

Actually, the expression of the time evolution of the mean is obtained as the sum of a term dependent of n_0 and a function $\mathcal{M}_1(1,t)$ independent of n_0:

$$\langle n(t)\rangle = E_\alpha\Bigg((\lambda-\mu)t^\alpha\Bigg)n_0 + \frac{\nu}{\lambda-\mu}\mathcal{M}_1(1,t) \ , \tag{29}$$

where

$$E_\alpha(z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+1)} \tag{30}$$

is the Mittag–Leffler function and

$$\mathcal{M}_1(1,t) = E_\alpha\Bigg((\lambda-\mu)t^\alpha\Bigg) - 1 \ , \tag{31}$$

where $\mathcal{M}_1(1,t) \to e^{(\lambda-\mu)t} - 1$ in the limit $\alpha \to 1$.

Then, the time dependent variance is expressed by the sum of functions independent of n_0 and that dependent of n_0. In the same way, the variance can have a steady state and a non-steady state component.

$$\sigma_n(t)^2 = \left(n_0 + \frac{\nu}{\lambda-\mu}\right)^2 \times \mathcal{M}_2(2,t)$$

$$+\left(\frac{\lambda+\mu}{\lambda-\mu}\right)\left\{n_0 + \frac{1}{2}\frac{\nu}{\lambda+\mu} + \frac{1}{2}\frac{\nu}{\lambda-\mu}\right\} \times \mathcal{M}_2(1,t)$$

$$+\frac{\nu}{2(\lambda-\mu)}\left(1 - \frac{\lambda+\mu}{(\lambda-\mu)}\right) \times \mathcal{M}_1(1,t)\ , \tag{32}$$

where

$$\mathcal{M}_2(1,t) = E_\alpha\Big(2(\lambda-\mu)t^\alpha\Big) - E_\alpha\Big((\lambda-\mu)t^\alpha\Big) \tag{33}$$

and

$$\mathcal{M}_2(2,t) = E_\alpha\Big(2(\lambda-\mu)t^\alpha\Big) - E_\alpha^2\Big((\lambda-\mu)t^\alpha\Big)\ . \tag{34}$$

The expression of the variance in Eq. (32) can be given in terms of the nonlinear evolution modes $\mathcal{M}_1(1,t)$, $\mathcal{M}_2(1,t)$ and $\mathcal{M}_2(2,t)$ with the Mittag–Leffler function. In the limit $\alpha \to 1$, $\mathcal{M}_2(1,t) \to e^{(\lambda-\mu)t}\Big(e^{(\lambda-\mu)t} - 1\Big)$ and $\mathcal{M}_2(2,t) \to 0$. The first term with n_0^2 dependence appears in Eq. (32). This is one of the special consequences due to the effect of long-memory in the FLBDP. The n_0^2 dependence in Eq. (32) disappears in the limit $\alpha \to 1$. In this way, the values of moments are expressed in terms of the Mittag–Leffler function. Namely, they are nonlinear functions of t^α, they cannot be described by a finite number of scaling functions ("multi-fractal" nature) (cf. References [29, 30, 32]).

One should note that depending on the values of the three parameters λ, μ and ν, there appear states of a steady state and of a non-steady state. In the cases $\mu - \lambda > 0$ and $\nu > 0$, when $t \to \infty$, the mean, variance and third order cumulant at the steady state are given by

$$\langle n(\infty)\rangle = \frac{\nu}{\mu-\lambda}\ , \sigma_n^2(\infty) = \frac{\nu\mu}{(\mu-\lambda)^2} \text{ and } \kappa_3(\infty) = \frac{(\lambda+\mu)\nu\mu}{(\lambda-\mu)^3}\ . \tag{35}$$

The ratio of the mean to the variance is called the Fano factor $F(\infty) = \frac{\sigma_n^2(\infty)}{\langle n(\infty)\rangle} > 1$. The negative polynomial distribution expresses the state of overdispersion.

The mean (29) and the variance (32) are expressed in terms of the Mittag–Leffler function. On account of the asymptotic nature of the Mittag–Leffler function, (i) $E_\alpha(-z) \approx 1 - z/\Gamma(1+\alpha)$, $z << 1$; (ii) $E_\alpha(-z) \approx \exp(-z/\Gamma(1-\alpha))$, $z \leq 1$ and (iii) $E_\alpha(-z) \approx z^{-1}/\Gamma(1-\alpha)$, $z >> 1$, one can infer the physical origin of the multi-fractal nature of the stochastic birth-death processes in Eq. (6).

4.2 Moments under critical condition

Although we have pointed out that the power law nature appears when considering the asymptotic property of the Mittag–Leffler function in Eqs. (29) and (32), the power law t^α nature can appear when the critical conditions are realized, as shown below.

When $\lambda = \mu$ and $\nu \neq 0$, the mean, the second moment and the variance are expressed as follows:

$$\langle n(t)\rangle = n_0 + \frac{\nu t^\alpha}{\Gamma(1+\alpha)}\ , \tag{36}$$

$$\langle n(t)^2\rangle = n_0^2 + \frac{(2(\lambda+\nu)n_0 + \nu)t^\alpha}{\Gamma(1+\alpha)} + \frac{2(\lambda+\nu)\nu t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{37}$$

and

$$\sigma_n^2(t) = \frac{(2n_0\lambda + \nu)t^\alpha}{\Gamma(1+\alpha)} + \left(\frac{2(\lambda+\nu)\nu}{\Gamma(1+2\alpha)} - \frac{\nu^2}{\Gamma(1+\alpha)^2}\right)t^{2\alpha}\ . \tag{38}$$

In this case, both the mean and the variance show the fractional power law. The variance has two types of scaling functions, t^α and $t^{2\alpha}$. Note that the variance depends on the initial value n_0. Further, the second part of the nonlinear modes $t^{2\alpha}$ does not vanish even when $\alpha \to 1$.

When $\lambda = \mu$ and $\nu = 0$, the above expressions reduce to

$$\langle n(t)\rangle = n_0, \langle n(t)^2\rangle = n_0^2 + \frac{2\lambda n_0}{\Gamma(1+\alpha)} t^\alpha \quad \text{and} \quad \sigma_n^2(t) = \frac{2\lambda n_0}{\Gamma(1+\alpha)} t^\alpha \ . \tag{39}$$

Interestingly, this case is equivalent to a fractional version of a critical Galton-Watson process [38]. The parameter set ($\lambda = \mu$ and $\nu = 0$) corresponds to the linear birth-death process due to Orsingher and Polito [31] for $n_0 = 1$. Clayton reported that his numerical simulation on 3D scroll waves [39, 40] as a model of ventricular fibrillation, indicates $\lambda \sim \mu$ and $\nu = 0$. The features of the temporal growth of the mean and variance in Eq. (39) will be discussed in a later section.

4.3 Extinction probability and waiting time distribution

The extinction probability and waiting time distribution are obtained in terms of the Mittag–Leffler function and the Meixner polynomials as

$$p_\alpha(0,t) = \left(1 - \frac{\lambda}{\mu}\right)^{\frac{\nu}{\lambda}} \sum_{n=0}^{\infty} \frac{M_n(n_0; \frac{\nu}{\lambda}, \frac{\lambda}{\mu})}{n!} \left(\frac{\lambda}{\mu}\right)^n E_\alpha\big(n(\lambda-\mu)t^\alpha\big) \tag{40}$$

and

$$f_\alpha(\tau) = \left(1 - \frac{\lambda}{\mu}\right)^{\frac{\nu}{\lambda}} \sum_{n=1}^{\infty} \frac{M_n(n_0; \frac{\nu}{\lambda}, \frac{\lambda}{\mu})}{n!} \left(\frac{\lambda}{\mu}\right)^n n\beta\tau^{\alpha-1} E_{\alpha,\alpha}\big(n(\lambda-\mu)\tau^\alpha\big) \ . \tag{41}$$

These expressions are valid for arbitrary initial value n_0 which is involved in the Meixner polynomials $M_n(n_0; \beta, c)$.

4.4 Waiting time distribution under critical condition

The generating function $\lambda = \mu$ under the constraint gives the exact extinction probability $p_\alpha(n,t)$ for n_0 from the generating function in Eq. (22) (i.e., $p_\alpha(0,t) = g_\alpha(z=0,t)$).

$$p_\alpha(0,t) = \sum_{n=0}^{n_0} \binom{n_0}{n} \int_0^\infty \exp(-\omega) E_{\alpha,1}^{n+\frac{\nu}{\lambda}}(-\omega\lambda t^\alpha)\ d\omega \ , \tag{42}$$

where $E_{\alpha,\beta}^\gamma(x)$ is the generalized Mittag–Leffler function [42] defined in Eq. (23).

The waiting time distribution is obtained by taking derivative with respect to time as

$$f_\alpha(\tau) = \lambda\tau^{\alpha-1} \sum_{n=1}^{n_0} \binom{n_0}{n} \int_0^\infty \omega \exp(-\omega) \tilde{E}_{\alpha,\alpha}^{n+\frac{\nu}{\lambda}}(-\omega\lambda\tau^\alpha)\ d\omega \ , \tag{43}$$

where $\tilde{E}_{\alpha,\beta}^\gamma(x)$ is a generalized Mittag–Leffler function defined by

$$\tilde{E}_{\alpha,\beta}^\gamma(x) = \sum_{n=0}^\infty \frac{\Gamma(\gamma+n+1)}{(n+1)!\Gamma(\gamma)\Gamma(\alpha n+\beta)}\ x^n \ . \tag{44}$$

(cf. the difference between Eqs. (23) and (44)) [42]

On account of the functional forms of the Meixner polynomials, one can analyse the features of the initial transients which is depending on the initial value n_0 in Eq. (25).

5 Parameter Estimation and Applications

5.1 Parameter estimation

The probability mass (PM) at the steady state for the master equation in Eq. (6) is given in Eq. (10). One can see that the two parameters

$$\theta_1 = \frac{\nu}{\lambda} \quad \text{and} \quad \theta_2 = \frac{\lambda}{\mu} \tag{45}$$

can be estimated by (i) the method of the least square fitting to observed PM, or (ii) the method of moments, i.e., the mean M and the variance V, since θ_1 and θ_2 are related to them as

$$M = \frac{\theta_1\theta_2}{1-\theta_2} \quad \text{and} \quad V = \frac{\theta_1\theta_2}{(1-\theta_2)^2}. \tag{46}$$

In taking the latter, the two parameters θ_1 and θ_2 are estimated by the observed values of M and V as

$$\theta_1 = \frac{M^2}{V-M} \quad \text{and} \quad \theta_2 = \frac{V-M}{V} \ . \tag{47}$$

Then, one can infer the three parameters (λ, μ, ν) as

$$\lambda = \frac{\beta\theta_2}{1-\theta_2}, \mu = \frac{\beta}{1-\theta_2} \text{ and } \nu = \frac{\beta\theta_1\theta_2}{1-\theta_2} \quad , \tag{48}$$

provided that the parameter,

$$\beta \equiv \mu - \lambda, \tag{49}$$

is inferred from the scaling exponent of the WTD for large τ.

It is clear that the parameters θ_1 and θ_2 are estimated from the steady state distribution in Eq. (10). Then, the estimation of β from the WTD in Eq. (41) gives the full three parameters (λ, μ, ν). These are the results for the Markovian limit. In the case of non-Markovian system with the present memory, the dominant term for large τ is the Mittag–Leffler function as

$$f_\alpha(\tau) \sim \tau^{\alpha-1} E_{\alpha,\alpha}(-\beta\tau^\alpha) \sim \tau^{-\alpha-1} \ (\tau \to \infty) \ .$$

Therefore, the procedure is almost the same as in the Markovian case.

5.2 Applications to some physical systems

5.2.1 Stochastic epidemic dynamics in SIRXY model

Stollenwerk and Jansen [16] studied evolution toward criticality in an epidemiological model based on SIR system with a mutant strain Y and severely affected hosts X. The full system of the SIRYX model is described by multiple-variables in the master equation. They showed that the marginal probability density $P(Y,t)$ reduces to the master equation in Eq. (1) with $\lambda \neq 0$, $\mu \neq 0$ and $\nu \neq 0$ without memory under the initial condition $n_0 = 1$. By taking $\nu \approx 0$, they obtained the expression of the mean as $\langle n(t)\rangle \approx \exp(-gt)$ and of the variance as $\sigma_n^2(t) \approx \frac{(2\gamma-g)}{g}(\exp(-gt) - \exp(-2gt))$ (Eq. (17) in their paper) under the assumption of the existence of steady states ($\mu > \lambda$) for X and Y.

Our model is just a fractional generalization of their SIRYX model. The mean and variance in Eqs. (29) and (32) are expected under the situation that a long time memory is relevant. Figure 1 shows the time evolutions of (i) the mean and (ii) the variance as a function of the fractional index of α. The mean simply decays with a power-law tail in the case of $\alpha < 1$. The variance has a bumped structure starting from 1 as a function of time, and has a power law decay as $t \to \infty$ for $\alpha < 1$. It is an interesting problem to examine a space-dependent SIRXY model and to compare with the present phenomenological prediction.

5.2.2 Dynamics of filaments in 3D scroll wave

Then let us consider the dynamics of filaments (phase-singularities, PS) in a 3D scroll wave system due to Clayton. [39, 40] He reported the generation rate $c(n) = c_1 n$ and the death/loss rate $a(n) = a_1 n$ based on 3D numerical simulations with the three-variables simplified ion channel model (3V-SIM) developed by Fenton-Karma [41]. It seems that the obtained rates of creation $c(n)$ and annihilation $a(n)$ of filaments are described by a critical Galton-Watson process [38] $\lambda \sim \mu$ and $\nu = 0$. In the case of a critical Galton-Watson process [38] with $\alpha = 1$ and $n_0 = 1$, the lifetime distribution takes the power law type,

$$f_1(t) = \lambda/(1+\lambda t)^2 \sim t^{-2} \quad \text{as} t \to \infty \ , \tag{50}$$

though the lifetime distribution of the number of filaments looks like exponential. Also, the counting number and its variance in Eq. (39) become

$$\langle n(t)\rangle = 1 \text{ and } \sigma_n^2(t) = 2\lambda t \tag{51}$$

When the system is critical $\lambda \sim \mu$, the mean is expected to be constant.

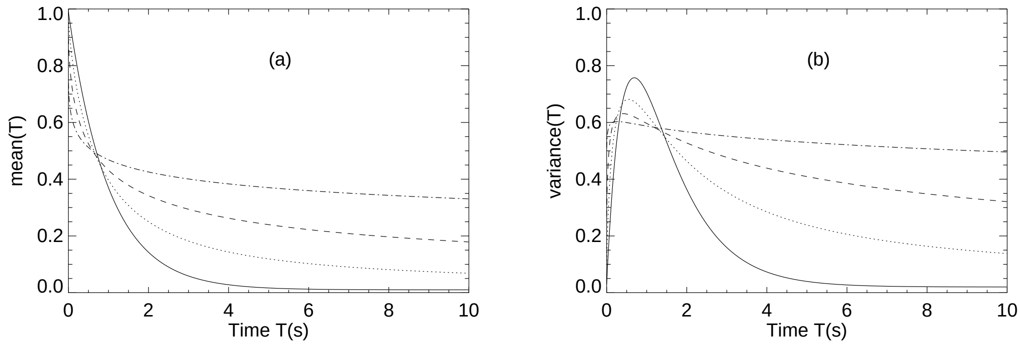

Figure 1: Effect of memory upon the stochastic SIRXY epidemic model. The features of time evolution of (a) the mean (29) and (b) the variance (32) starting from $n_0 = 1$ are depicted as a function of α with the parameters $(\lambda = 1, \mu = 2, \nu = 0.01)$: (i) $\alpha = 1$ (solid line); (ii) $\alpha = 0.75$ (dotted line); (iii) $\alpha = 0.5$ (dashed line); (iv) $\alpha = 0.25$ (dash-dot line). As α decreases, the relaxation time to the equilibrium state becomes slower, as seen in the figures.

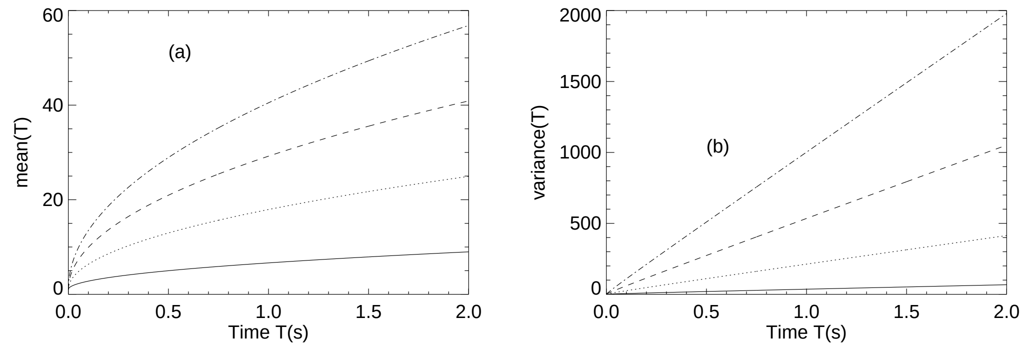

Figure 2: Effect of memory upon stochastic FLBDP model under the critical condition $\lambda = \mu$. The features of time evolution of (a) the mean and (b) the variance starting from $n_0 = 1$ are depicted as a function of ν with the parameters $(\lambda = 1, \alpha = 0.5)$: (i) $\nu = 5.0$ (solid line); (ii) $\nu = 15.0$ (dotted line); (iii) $\nu = 25.0$ (dashed line); (iv) $\nu = 35.0$ (dash-dot line). As ν increases, the linear growth becomes dominant as seen in the figures.

Since the time evolution of the mean value is observed (Fig.5b in [40]) and it tends to saturate, let us assume $0 < \alpha < 1$. The lifetime distribution in Eq. (44) with $n_0 = 1$ and $\nu = 0$ leads to a simple form with a power-law tail as

$$f_\alpha(\tau) = \lambda \tau^{\alpha-1} \int_0^\infty e^{-\omega} \omega E_{\alpha,\alpha}(-\lambda w \tau^\alpha)\, d\omega \sim \tau^{-\alpha-1}. \tag{52}$$

When we assume that ν is a small positive number and $\lambda = \mu$, the mean and variance diverge, but the speed of divergence of them slows down as

$$\langle n(t) \rangle = 1 + \frac{\nu t^{1/2}}{\Gamma(3/2)}\ , \sigma_n^2(t) = (2\lambda + \nu) \frac{\nu t^{1/2}}{\Gamma(3/2)} + \left(2 \frac{(\lambda + \nu)\nu}{\Gamma(2)} - \frac{\nu^2}{\Gamma(3/2)^2} \right) t\ . \tag{53}$$

The long-time tail of the waiting time distribution becomes $f_{1/2}(\tau) \sim \tau^{-3/2}$ as $\tau \to \infty$. In the Markovian limit, the mean and variance in Eqs. (36) and (38) reduce to

$$\langle n(t) \rangle = 1 + \nu t \quad \text{and} \quad \sigma_n^2(t) = (2\lambda + \nu)\nu t + \lambda \nu t^2\ . \tag{54}$$

It seems that the linear t and the square t dependence for the mean and variance is not consistent with his experiment. Figure 2 shows the time evolution of the mean and variance as a function of ν with $\alpha \sim 1/2$. A problem in the above assumptions is that one must take $\nu > 0$ to fit the evolution of the mean. To get a good fit to the total features of his numerical experiment and to have a plausible explanation of the long lasting PS trajectory in Fig.8 [40], extensive analyzes are required.

5.2.3 QCD dynamics

Suzuki and Biyajima [20] studied the fractional Fokker-Planck equation associated with the fractional master equation in Eq. (6) under the initial condition $n_0 = 0$. Since the initial condition may be uncontrollable, we think that the results and discussions under the initial condition $n_0 = 0$ is not necessarily desirable in comparing the real experiments on QCD. Provided that the initial value obeys, for example, the Poisson distribution with the parameter θ, $P(n_0) = \frac{\theta^{n_0}}{n_0!}e^{-\theta}$, the generating function should be averaged over it, the averaged generating function $\overline{g_\alpha[z,s]} \equiv \sum_{n_0=0}^{\infty} P(n_0) g_\alpha[z,s|n_0]$ in the Laplace domain becomes

$$\overline{g_\alpha[z,s]} = \frac{zs^{\alpha-1}}{\lambda z - \mu}\int_0^1 dx(1-xz)^{-\frac{\nu}{\lambda}}\left[1 - \frac{(\lambda-\mu)xz}{\lambda z - \mu}\right]^{\frac{\nu}{\lambda}+\frac{s^\alpha}{\lambda-\mu}-1} \times \exp\left[-\frac{(1-z)\theta}{1-xz}\right]. \tag{55}$$

The expression of the generating function in the Laplace domain in Eq. (21) is expedient to study the effect of initial condition dependence. On the other hand, it is not convenient in taking summation over the initial distribution $P(n_0)$ in the time domain in Eq. (25).

The expressions of the mean and variance under the Poisson distribution of n_0 with the parameter θ by the generating function in Eq. (55) are identical with those by the replacements: $n_0 \to \theta$ and $n_0^2 \to \theta(\theta+1)$ in Eqs. (29) and (32). The features of time evolution for (a,b) $n_0 = 0$, (c,d) $n_0 = 1$ and (e,f) $P(n_0|\theta = 3)$ are displayed in Fig. 3. One can see from these figures that the growth rate of the mean value is suppressed as the value of α decreases. On the other hand, the growth rate of evolution of the variance increases as the initial value n_0 decreases. The detailed analysis of the initial dependence is important for complex systems in the real world when the initial condition is uncontrollable.

6 Summary and Remarks

In this paper, a fractional generalized birth-death process with a linear birth and a death rate in Eq. (6) was studied based on the master equation. The main features of the present fractional linear birth-death process are summarized as follows: The generating function is solved exactly in the integral form though the relevant differential equation being classified into the Heun's differential equation with an accessory parameter $q \neq 0$ in Eq. (13). Its solution is also rewritten by the combination of the Mittag–Leffler function (MLF) with the coefficients of the Meixner orthogonal polynomials. The analytical expression in terms of the Meixner polynomials is quite useful to get the expressions of the time-dependent moments and the waiting time (lifetime) distribution. The fractional linear birth-death process (FLBDP) in Eq. (6) can cover various processes which have been studied previously as remarked in the introduction.

Since the effect of memory of the model in Eq. (6) is introduced by the fractional derivative in Eq. (8), one may think that applicability of it is quite limited. Interestingly, we can observe many examples in real experiments and numerical simulations where the model in Eq. (6) is available on account of the universal power-law scaling in time and frequency domains. In the master equation approach, the effect of a long-memory appears in the waiting time distribution (WTD) as a power-law tail of the MLF. When the steady state is realized (e.g., $\mu > \lambda$ and $\nu > 0$), the correlation function (CF) can be also expressed in terms of a class of the MLFs. The α dependence of the MLF can be expected from Fig.1 (a) in this paper. In the Fokker-Planck approximation of Eq. (6), expression of the CF can be obtained as a sum of the MLFs even when the birth-death rates are nonlinear function of n like $\lambda_n = c_0 + c_1 n + c_2 n^2$ and $\mu_n = a_1 n + a_2 n^2$.

Davidsen and coworkers [43, 44] studied the statistical features of (i) a state of amplitude turbulence and (ii) a state of negative filament tension. They reported the creation and the death rates as $\lambda_n = \lambda$ and $\mu_n = k_2 n^2$ in the amplitude wave turbulence [44] of 3D CGLE. Also, the rates are linear: $\mu_n = \mu n + \epsilon$ and $\lambda_n = \lambda n + \nu$ in the state of negative filament tension [43, 44] for 3D CGLE and 3D Barkley's model. As far as the distribution of the number of singularities is concerned, it seems that the Markovian description can catch the feature of their numerical experiments. However, the simple death rate ϵ was estimated as negative values [43] and [44]. In the case of the negative dissipation, the diffusion coefficient takes a negative value in the Fokker-Planck equation associated with the master equation in Eq. (6) under the Poisson transform [20]. The detailed analyzes on these cases are out of scope in the present theoretical method of the FLBDP.

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Appendix A

Derivation of Eq. (21)

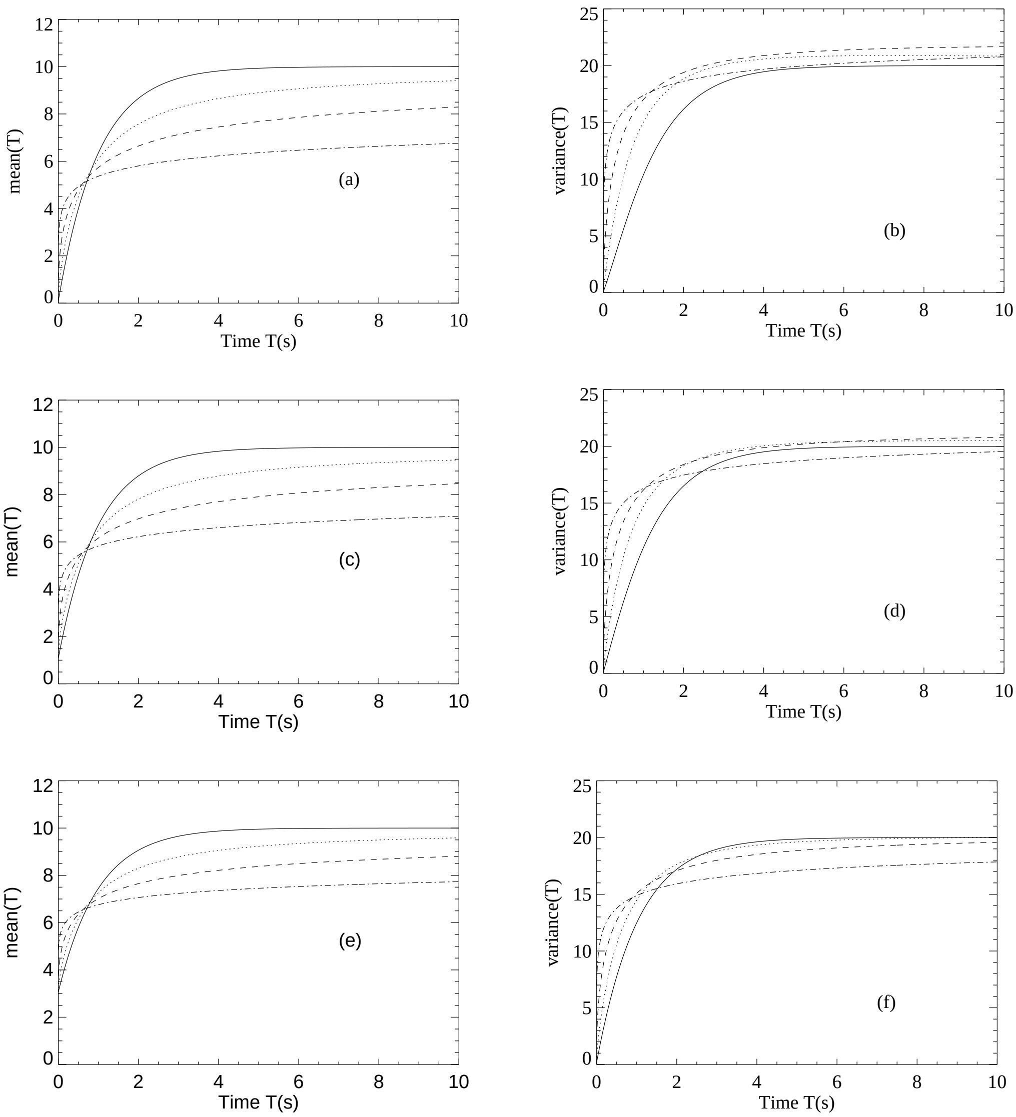

Figure 3: Effect of the initial condition on the feature of time evolution of the mean and variance starting from the initial value (a,b) $n_0 = 0$, (c,d) $n_0 = 1$ and (e,f) $P(n_0|\theta = 3)$: (i) $\alpha = 1$ (solid line); (ii) $\alpha = 0.75$ (dotted line); (iii) $\alpha = 0.50$ (dashed line); (iv) $\alpha = 0.25$ (dash-dot line). The other parameters $\lambda = 1, \mu = 2.0, \nu = 10.0$ take the same values in the figures. The features of anomalous fluctuations in the transient state are suppressed as the initial value n_0 increases under the existence of memory with $0 < \alpha < 1$.

The direct integration of the Laplace transformed generating function in Eq. (12) leads to Eq. (18). It is convenient to take a variable transformation as

$$1-x=\frac{\omega}{z}\cdot\frac{z-1}{\omega-1}\ . \tag{A1}$$

In this case, Eq. (A1) gives

$$\omega=\frac{(1-x)z}{1-xz}\ ,\ 1-\omega=\frac{(1-z)}{1-xz} \tag{A2}$$

and

$$dx=-\frac{1-z}{z}\frac{1}{(1-\omega)^2}d\omega\ , \tag{A3}$$

which brings Eq. (18) to the expression of the generating function $g_\alpha[z,s]$ to Eq. (21):

$$g_\alpha[z,s]=s^{\alpha-1}\frac{z^{n_0+1}}{(\lambda z-\mu)}\int_0^1(1-x)^{n_0}(1-xz)^{-n_0-\frac{\nu}{\lambda}}\times\left[1-\frac{(\lambda-\mu)xz}{\lambda z-\mu}\right]^{\frac{\nu}{\lambda}+\frac{s^\alpha}{\lambda-\mu}-1}dx. \tag{A4}$$

Note here that $\sum_{n=0}^{\infty}p_\alpha[n,s]=g_\alpha[z=1,s]=\frac{1}{s}$. It should be noted that $g_\alpha[z,s]=zL_\alpha[z,s]$, where $L_\alpha[z,s]$ is the integral representation of the general Heun's function.

When $\mu=0$, Eq. (A4) reduces to the generating function of the fractional generalized birth process (FGBP) [32] as

$$g_\alpha[z,s]=z^{n_0}\left(\frac{s^{\alpha-1}}{\lambda}\right)\int_0^1(1-x)^{\frac{s^\alpha}{\lambda}+n_0+\frac{\nu}{\lambda}-1}(1-xz)^{-(n_0+\frac{\nu}{\lambda})}dx\ , \tag{A5}$$

$$=z^{n_0}\left(\frac{s^{\alpha-1}}{s^\alpha+n_0\lambda+\nu}\right)F\left(\left[n_0+\frac{\lambda}{\nu},1\right],\left[\frac{s^\alpha}{\lambda}+n_0+\frac{\nu}{\lambda}+1\right];z\right), \tag{A6}$$

where $F([a,b],[c];z)$ is Euler's integral representation of the Gauss hypergeometric function defined by

$$F([a,b],[c];z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx\ , \tag{A7}$$

where $\Gamma(z)$ is the Gamma function: $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$. The integral representation is quite useful for obtaining the expressions of the moments, the extinction probability, the waiting time distribution and so on.

Appendix B

Derivation of Eq. (22)

When $\lambda=\mu$, the generating function in the Markovian limit becomes

$$g_1(z,t)=\left(1-\frac{1-z}{1+(1-z)\lambda t}\right)^{n_0}(1+(1-z)\lambda t)^{-\frac{\nu}{\lambda}} \tag{B1}$$

With the use of the binomial expansion, one obtains

$$g_1(z,t)=\sum_{n=0}^{n_0}\binom{n_0}{n}\frac{(z-1)^n}{[1+(1-z)\lambda t]^{n+\frac{\nu}{\lambda}}} \tag{B2}$$

Applying the Levy transform $g_\alpha[z,s]=\int_0^\infty s^{\alpha-1}\exp(-s^\alpha\tau)g_1(z,\tau)d\tau$ to (B2), one obtains

$$g_\alpha[z,s]=\sum_{n=0}^{n_0}\binom{n_0}{n}(z-1)^n\int_0^\infty s^{\alpha-1}\frac{\exp(-s^\alpha t)\ dt}{[1+(1-z)\lambda t]^{n+\frac{\nu}{\lambda}}} \tag{B3}$$

$$=\sum_{n=0}^{n_0}\binom{n_0}{n}(z-1)^n s^{\alpha-1}\frac{(s^\alpha)^{b-1}\exp(\frac{s^\alpha}{a})}{a^b}\Gamma\left(1-b,\frac{s^\alpha}{a}\right)\ , \tag{B4}$$

where

$$a=\lambda(1-z)\ ,\quad b=n+\frac{\nu}{\lambda} \tag{B5}$$

and $\Gamma(z,p)$ is the incomplete Gamma function of the second kind,

$$\Gamma(z,p) = \int_p^\infty \exp(-y) y^{z-1}\ dy\ . \tag{B6}$$

Eq. (B4) is rewritten by putting $y = p+\omega$ in Eq. (B6) as

$$g_\alpha[z,s] = \sum_{n=0}^{n_0} \binom{n_0}{n} (z-1)^n \int_0^\infty d\omega \frac{\exp(-\omega) s^{\alpha b-1}}{(s^\alpha + a\omega)^b}\ . \tag{B7}$$

On account of the integral formulae of the generalized Mittag-Leffler function (Eq. (11.8) in Ref. [42]),

$$\int_0^\infty \exp(-st) t^{\beta-1} E^\gamma_{\alpha,\beta}(-a\omega t^\alpha) dt = s^{-\beta}(1 + a\omega s^{-\alpha})^{-\gamma}\ , \tag{B8}$$

we have the generating function in Eq. (22):

$$g_\alpha(z,t) = \sum_{n=0}^{n_0} \binom{n_0}{n} (z-1)^n \int_0^\infty d\omega \exp(-\omega) E^b_{\alpha,1}(-a\omega t^\alpha)\ . \tag{B9}$$

This is the simplest derivation of Eq. (22). The direct derivation from Eq. (21) is avoided due to its laborious calculation.

Appendix C

Derivation of Eq. (25)

The generating function in the Markovian limit is given by

$$g_1(z,t) = \left(1 + \frac{\lambda}{\lambda-\mu}(1-z)\left[e^{(\lambda-\mu)t} - 1\right]\right)^{-\frac{\nu}{\lambda}-n_0}$$
$$\times \left(1 + \frac{\lambda}{\lambda-\mu}(1-z)\left[e^{(\lambda-\mu)t} - 1\right] - (1-z)e^{(\lambda-\mu)t}\right)^{n_0}\ . \tag{C1}$$

The equation can be rewritten in the form:

$$g_1(z,t) = \left(\frac{\lambda z-\mu}{\lambda-\mu}\right)^{-\frac{\nu}{\lambda}} \left[1 - \frac{\lambda}{\lambda z-\mu}(z-1)e^{(\lambda-\nu)t}\right]^{-\frac{\nu}{\lambda}-n_0} \times \left[1 - \frac{\mu}{\lambda z-\mu}(z-1)e^{(\lambda-\mu)t}\right]^{n_0}\ . \tag{C2}$$

By applying the expansion in terms of the Meixner polynomials,

$$\sum_{n=0}^\infty \frac{M_n(x;\beta,c)}{n!}\theta^n = \left(1-\frac{\theta}{c}\right)^x \left(1-\theta\right)^{-x-\beta}\ , \tag{C3}$$

we obtain

$$g_1(z,t) = \left(\frac{\lambda z-\mu}{\lambda-\mu}\right)^{-\frac{\nu}{\lambda}} \sum_{n=0}^\infty \frac{M_n(n_0, \frac{\nu}{\lambda}, \frac{\lambda}{\mu})}{n!} \left(\frac{\lambda(z-1)}{\lambda z-\mu}\right)^n e^{n(\lambda-\mu)t}\ , \tag{C4}$$

where $M_n(x;\beta,c)$ is the Meixner polynomials of the first kind defined by

$$M_n(x;\beta,c) = (\beta)_n F\left([-x,-n],[\beta], 1-\frac{1}{c}\right), \tag{C5}$$

where $F([\alpha,\beta],[\gamma],x)$ is the hypergeometric function.

The extinction probability is obtained in the form:

$$p_1(0,t) = \left(\frac{\nu}{\mu-\lambda}\right)^{-\frac{\nu}{\lambda}} \sum_{n=0}^\infty \frac{M_n(n_0; \frac{\nu}{\lambda}, \frac{\lambda}{\mu})}{n!} \left(\frac{\lambda}{\mu}\right)^n e^{n(\lambda-\mu)t}\ . \tag{C6}$$

The waiting time distribution (or the lifetime distribution) is given by

$$f_1(\tau) = \left(\frac{\nu}{\mu-\lambda}\right)^{-\frac{\nu}{\lambda}} \sum_{n=0}^\infty \frac{M_n(n_0; \frac{\nu}{\lambda}, \frac{\lambda}{\mu})}{n!} \left(\frac{\lambda}{\mu}\right)^n n(\mu-\lambda)e^{n(\lambda-\mu)\tau}\ . \tag{C7}$$

The generating function for the fractional master equation is given by the Lévy transform [45, 46, 30] as

$$g_\alpha(z,t) = \left(\frac{\lambda z-\mu}{\lambda-\mu}\right)^{-\frac{\nu}{\lambda}} \sum_{n=0}^\infty \frac{M_n(n_0, \frac{\nu}{\lambda}, \frac{\lambda}{\mu})}{n!} \left(\frac{\lambda(z-1)}{\lambda z-\mu}\right)^n E_\alpha\big(n(\lambda-\mu)t^\alpha\big)\ . \tag{C8}$$

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