ON THE RANGE OF SELF-INTERACTING RANDOM WALKS ON AN INTEGER INTERVAL

By

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Abstract. We consider the range of a one-parameter family of self-interacting walks on the integers up to the time of exit from an interval. We derive the weak convergence of an appropriately scaled range. We show that the distribution functions of the limits of the scaled range satisfy a certain class of de Rham's functional equations. We examine the regularity of the limits.

1 Introduction

The range of random walk has been studied for a long time. Examining the range at the time the random walk leaves an interval is a simple and natural concern. Recently, Athreya, Sethuraman and Tóth [1] considered questions of this kind. They studied the range, local times and periodicity or "parity" statistics of some nearest-neighbor Markov random walks up to the time of exit from an interval of N sites. They derived several associated scaling limits as $N \to \infty$ and related the limits to various notions such as the entropy of an exit distribution, generalized Ray-Knight constructions, and Bessel and Ornstein-Uhlenbeck square processes.

Inspired by [1], we consider the ranges of a certain class of self-interacting random walks up to the time of exit from an interval. The study of self-interacting walks originated from the modeling of polymer chains in chemical physics. There are various models in this study. We consider the model defined by Denker and Hattori [2], Hambly, Hattori and Hattori [4], Hattori and Hattori [5], [6]. They constructed a natural one-parameter family of self-repelling and self-attracting walks on $\mathbf{Z}$ and the infinite pre-Sierpiński gasket. It interpolates

continuously between self-avoiding walk and simple random walk in the sense of exponents.

In general, most of the studies of self-interacting walks are difficult due to the lack of Markov property, even if they are one-dimensional. In the studies of Markov walks, we can use techniques in analysis, especially, potential theory. However, in the case of non-Markov walks, we cannot use most of the techniques used in the studies of Markov walks. Most of the arguments in [1] depend heavily on the Markov property. Therefore, we have to use alternative methods for our study. We apply a recent result by the author [7] which considers a certain class of de Rham's functional equations.

Now we state our settings and results briefly. Let W_∞ be the path space of the nearest-neighbor walk starting at 0 on $\mathbf{Z}$. Let $\{P^u\}_{u \geq 0}$ be a one-parameter family of probability measures on W_∞ defined by [2] and [6]. We will give the precise definitions of them in Section 2. P^0 defines the self-avoiding walk on $\mathbf{Z}$ and P^1 defines the standard simple random walk. If $u \neq 1$, P^u defines a non-Markov random walk on $\mathbf{Z}$.

DEFINITION 1.1.   Let $n \in \mathbf{N} = \{1, 2, \ldots\}$ and $\omega \in W_\infty$. Let $R_n(\omega)$ be the range of ω up to the time of exit from $\{-2^n, \ldots, 2^n\}$, that is,

$R_n(\omega) =$ (the number of points which ω visits before it hits the points $\{\pm 2^n\}$).

Note that $2^n \leq R_n \leq 2^{n+1} - 1$.

Then, we have the following results which are analogous to [1], Proposition 2.1.

THEOREM 1.2.   (1) *Let* $u \geq 0$. *Then, the random variables* $\{(R_n/2^n) - 1\}_n$ *converges weakly to a distribution function* f_u *on* $[0, 1]$, $n \to \infty$.

(2) *Let* $u > 0$. *Then* f_u *satisfies a certain class of de Rham's functional equations* [3]:

$$f(x) = \begin{cases} \Phi(A_{u,0}; f(2x)) & 0 \leq x \leq 1/2 \\ \Phi(A_{u,1}; f(2x - 1)) & 1/2 \leq x \leq 1, \end{cases} \tag{1.1}$$

where we let

$$\Phi(A; z) = \frac{az + b}{cz + d} \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and,}$$

$$A_{u,0} = \begin{pmatrix} x_u & 0 \\ -u^2 x_u^2 & 1 \end{pmatrix}, \quad A_{u,1} = \begin{pmatrix} 0 & x_u \\ -u^2 x_u^2 & 1 - u^2 x_u^2 \end{pmatrix}, \quad x_u = \frac{2}{1 + \sqrt{1 + 8u^2}}.$$

(3) *Let $\tilde{P}^u$ be the probability measure on $[0,1]$ such that its distribution function is f_u. If $u = 1$, $\tilde{P}^u$ is absolutely continuous with respect to the Lebesgue measure on $[0,1]$. If $u \neq 1$, $\tilde{P}^u$ is singular.*

We remark that $\tilde{P}^0 = \tilde{P}_n^0 = \delta_{\{0\}}$, where δ denotes a point mass.

Let us denote the Hausdorff dimension of $K \subset [0,1]$ by $\dim_H(K)$. Let us define the Hausdorff dimension of a probability measure μ on $[0,1]$ by $\dim_H \mu = \inf\{\dim_H(K) : K \in \mathscr{B}([0,1]), \mu(K) = 1\}$. Let $s(p) = -p \log p - (1-p)\log(1-p)$ for $p \in [0,1]$.

If $0 < u < \sqrt{3}$, $(A_{u,0}, A_{u,1})$ satisfies the conditions (A1)–(A3) in [7], so we can apply the results in [7] to this case and obtain the following results. We refer the reader to [7] for details.

THEOREM 1.3.   (1) *If $u \neq 1$ and $0 < u < \sqrt{3}$, then $\dim_H \tilde{P}^u < 1$.*
(2) *If $0 < u < 1$, then $\dim_H \tilde{P}^u \leq s(x_u)/\log 2$. Moreover, $\tilde{P}^u(K) = 0$ for any Borel set K with $\dim_H(K) < s(2x_u/(1 + x_u))/\log 2$.*

We also examine whether $\tilde{P}^u$ has atoms.

THEOREM 1.4.   (1) *Let $u \leq \sqrt{3}$. Then, $\tilde{P}^u$ has no atoms.*
(2) *Let $u > \sqrt{3}$. Then, $\tilde{P}^u(\{x\}) > 0$ for any $x \in D \cap (0,1]$. Here D is the set of dyadic rationals on $[0,1]$.*

In Section 2, we will describe the settings. In Section 3, we will show Theorem 1.2 and Theorem 1.4.

2   Preliminaries

We briefly state our settings by following [2] and [5]. See the references for details.

For each $n \in \mathbf{N} \cup \{0\}$, let

$$W(n) = \{(\omega(0), \omega(1) \cdots \omega(n)) \in \mathbf{Z}^{n+1} : \omega(0) = 0,$$
$$|\omega(i) - \omega(i+1)| = 1, 0 \leq i \leq n-1\}.$$

Let $W^* = \bigcup_{n=0}^{\infty} W(n)$. Let $L(\omega) = n$ for $\omega \in W(n)$. For $\omega \in W^*$, we define $T_i^M(\omega)$, $i, M \in \mathbf{N} \cup \{0\}$, by $T_0^M(\omega) = 0$,

$$T_i^M(\omega) = \min\{j > T_{i-1}^M(\omega) : \omega(j) \in 2^M \mathbf{Z} \setminus \{\omega(T_{i-1}^M(\omega))\}\}, \quad i \geq 1.$$

Let $T_i^M(\omega) = +\infty$ if the above minimum does not exist.

We define a decimation map $Q_M : W^* \to W^*$, $M \in \mathbf{N}$, by $(Q_M \omega)(i) = \omega(T_i^M(\omega))$ for i such that $T_i^M(\omega) < +\infty$. Let Q_0 be the identity map on W^*. Let $(2^{-M} Q_M \omega)(i) = 2^{-M} \omega(T_i^M(\omega))$. Then, $2^{-M} Q_M \omega \in W^*$ and $L(2^{-M} Q_M \omega) = k$, where $k = \max\{i : T_i^M(\omega) < \infty\}$. Let $W_{N,+(resp.-)} = \{\omega \in W^* : L(\omega) = T_1^N(\omega), \omega(T_1^N(\omega)) = +(resp.-)2^N\}$ and $W_N = W_{N,+} \cup W_{N,-}$.

For $\omega \in W_{N+n,+}$, let $\omega' = 2^{-N} Q_N \omega$. For $1 \leq j \leq L(\omega')$, we let $\omega_j = (0, \omega(T_{j-1}^N(\omega) + 1) - \omega(T_{j-1}^N(\omega)), \ldots, \omega(T_j^N(\omega)) - \omega(T_{j-1}^N(\omega))) \in W_N$, and, $\tilde{\omega}_j = \text{sign}(\omega(T_j^N(\omega)) - \omega(T_{j-1}^N(\omega)))\omega_j \in W_{N,+}$.

Now we will define a probability measure $P_{N,\pm}^u$, $u \geq 0$, on $W_{N,\pm}$ by induction on N in the following manner. We recall that $x_u = 2/(1 + \sqrt{1 + 8u^2})$. Let $P_{1,+}^u(\{\omega\}) = u^{L(\omega)-2} x_u^{L(\omega)-1}$, $\omega \in W_{1,+}$, where we adopt the conventions $0^0 = 1$ and $0^n = 0$, $n \geq 1$. For $\omega \in W_{N+1,+}$, let

$$P_{N+1,+}^u(\{\omega\}) = P_{1,+}^u(\{\omega'\}) \prod_{i=1}^{L(\omega')} P_{N,+}^u(\{\tilde{\omega}_i\}). \tag{2.1}$$

We define $P_{N,-}^u(\{\omega\}) = P_{N,+}^u(\{-\omega\})$ for $\omega \in W_{N,-}$, $N \in \mathbf{N}$. Let P_N^u be a probability measure on W_N given by $P_N^u = (P_{N,+}^u + P_{N,-}^u)/2$.

We denote the set of the paths of infinite length by

$$W_\infty = \{(\omega(0), \omega(1), \ldots) \in \mathbf{Z}^{\mathbf{N} \cup \{0\}} : \omega(0) = 0, |\omega(i) - \omega(i+1)| = 1. i \geq 0\}.$$

Let the σ-algebra on this set be the family of subsets which is generated by cylinder sets. By [2], Proposition 2.5, there exists a probability measure P^u on W_∞ such that

$$P^u(\{\omega \in W_\infty : \omega(j) = \tilde{\omega}(j), 0 \leq j \leq L(\tilde{\omega})\}) = \frac{1}{2} P_{N,+(resp.-)}^u(\{\tilde{\omega}\}),$$

for any $\tilde{\omega} \in W_{N,+(resp.-)}$, $N \geq 1$.

3 Range of Random Walk on the Interval $[-2^n, 2^n]$ and its Scaling Limit

Here and henceforth, we assume that $u > 0$.

First we will show Theorem 1.2. The main ingredient of the proof is to show that $g_u(k/2^n) := P_{n,+}^u(R_n \leq 2^n + k - 1)$ satisfies (1.1) on the dyadic rationals. This

depends heavily on the definition of $P^u_{n,+}$ in Section 2. Then, we will see that the right continuous modification of g_u satisfies (1.1) on $[0,1]$. Next, we will show that the distribution of $R_n/2^n - 1$ converges to g_u weakly as $n \to \infty$ and examine the regularity of g_u.

We remark that $P^u(R_n = 2^n + k) = P^u_{n,+}(R_n = 2^n + k)$, $0 \le k \le 2^n$, $n \ge 1$.

LEMMA 3.1.

$$P^u_{N,+}\left(\frac{R_N}{2^N} - 1 \ge \frac{k}{2^n}\right) = P^u_{n,+}\left(\frac{R_n}{2^n} - 1 \ge \frac{k}{2^n}\right),$$

for any $N \ge n$, $0 \le k \le 2^n$ and $n \ge 1$.

PROOF.   Let $N > n$. Then,

$$P^u_{N,+}\left(\frac{R_N}{2^N} - 1 \ge \frac{k}{2^n}\right) = P^u_{N,+}(\{\omega \in W_{N,+} : \omega \text{ hits the point } \{-2^{N-n}k\}\})$$

$$= P^u_{N,+}(\{\omega : Q_{N-n}\omega \text{ hits the point } \{-2^{N-n}k\}\})$$

$$= P^u_{N,+}(\{\omega : 2^{-(N-n)}Q_{N-n}\omega \text{ hits the point } \{-k\}\})$$

$$= P^u_{n,+}(\{\zeta \in W_{n,+} : \zeta \text{ hits the point } \{-k\}\})$$

$$= P^u_{n,+}\left(\frac{R_n}{2^n} - 1 \ge \frac{k}{2^n}\right),$$

where in the fourth equality we have used [2] Proposition 2.2.   □

DEFINITION 3.2.   (1) Let g_u be a function on D given by $g_u((k+1)/2^n) = P^u_{n,+}(R_n \le 2^n + k)$, $-1 \le k \le 2^n - 1$. By Lemma 3.1, this is well-defined. We immediately see that $g_u(x)$ is increasing and $g_u(0) = 0$, $g_u(1) = 1$.

(2) Let $\tilde{g}_u$ be a function on $[0,1]$ given by $\tilde{g}_u(x) = \lim_{y \in D, y>x, y \to x} g_u(y)$, $0 \le x < 1$ and $\tilde{g}_u(1) = 1$. This is right continuous.

The following is a key proposition.

PROPOSITION 3.3.   *The function g_u satisfies (1.1) on D, that is,*

$$P^u_{n+1,+}(R_{n+1} \le 2^{n+1} + k) = \begin{cases} \Phi(A_{u,0}; P^u_{n,+}(R_n \le 2^n + k)) & -1 \le k \le 2^n - 1 \\ \Phi(A_{u,1}; P^u_{n,+}(R_n \le k)) & 2^n - 1 \le k \le 2^{n+1} - 1. \end{cases}$$

PROOF. If $k = -1$, we have that $\Phi(A_{u,0}; P^u_{n,+}(R_n \le 2^n + k)) = \Phi(A_{u,0}; 0)$ $= 0 = P^u_{n+1,+}(R_{n+1} \le 2^{n+1} + k)$. If $k = 2^n - 1$, we have that $\Phi(A_{u,0}; P^u_{n,+}(R_n \le 2^n + k)) = \Phi(A_{u,0}; 1) = \Phi(A_{u,1}; 0) = \Phi(A_{u,1}; P^u_{n,+}(R_n \le k))$. Then, it is sufficient to show this assertion in the following two cases. For any $\omega \in W_{n+1,+}$, define $(\omega', \tilde{\omega}_1, \ldots, \tilde{\omega}_{L(\omega')})$ as in Section 2.

Case 1. $0 \le k \le 2^n - 1$. We have

$$P^u_{n+1,+}(R_{n+1} \le 2^{n+1} + k) = \sum_{m=1}^{\infty} P^u_{n+1,+}(\{\omega : L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\}).$$

Since $0 \le k \le 2^n - 1$, we see that $\omega' \in W_{1,+}$ does not hit -1 for any $\omega \in W_{n+1,+}$ with $R_{n+1}(\omega) \le 2^{n+1} + k$. Then we see that

$$\{\omega : L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\}$$
$$= \{\omega : \omega' = (0, 1, 0, 1, \ldots, 0, 1, 2), L(\omega') = 2m,$$
$$R_n(\tilde{\omega}_{2i-1}) \le 2^n + k, 1 \le i \le m\}.$$

By (2.1), we see that

$$P^u_{n+1,+}(\{\omega : L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\})$$
$$= P^u_{1,+}(\{\zeta : \zeta = (0, 1, 0, 1, \ldots, 0, 1, 2), L(\zeta) = 2m\}) \cdot P^u_{n,+}(R_n \le 2^n + k)^m.$$
$$= u^{2m-2} x_u^{2m-1} P^u_{n,+}(R_n \le 2^n + k)^m.$$

Then,

$$P^u_{n+1,+}(R_{n+1} \le 2^{n+1} + k) = \sum_{m=1}^{\infty} u^{2m-2} x_u^{2m-1} P^u_{n,+}(R_n \le 2^n + k)^m$$
$$= \Phi(A_{u,0}; P^u_{n,+}(R_n \le 2^n + k)),$$

which is the desired result.

Case 2. $2^n \le k \le 2^{n+1} - 1$.

Since $L(\omega') = 2m$, we can write $\omega' = (0, \varepsilon_1, 0, \varepsilon_2, \ldots, 0, \varepsilon_{m-1}, 0, 1, 2)$, $\varepsilon_i \in \{\pm 1\}$, $1 \le i \le m - 1$. Then we see that

$$\{\omega : L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\}$$
$$= \bigcup_{i=0}^{m-1} \{\omega : \sharp(j : \varepsilon_j = -1) = i, L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\}.$$

We remark that the union in the above is disjoint.

For $1 \le i \le m - 1$,

$$\{\omega : \sharp(j : \varepsilon_j = -1) = i, L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\}$$

$$= \bigcup_{1 \le n_1 < n_2 < \cdots < n_i \le m-1} \{\omega : \{j : \varepsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\},$$

$$L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\}.$$

We remark that the union in the above is disjoint.
By (2.1),

$$P^u_{n+1,+}(\{\omega : \{j : \varepsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\},$$

$$L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\})$$

$$= P^u_{n+1,+}(\{\omega : \{j : \varepsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\},$$

$$L(\omega') = 2m, R_n(\tilde{\omega}_{2n_j}) \le k, 1 \le j \le i\})$$

$$= P^u_{1,+}(\{\omega' : \{j : \varepsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\},$$

$$L(\omega') = 2m\}) P^u_{n,+}(R_n \le k)^i$$

$$= u^{2m-2} x_u^{2m-1} (P^u_{n,+}(R_n \le k))^i.$$

Since the number of choices $\{n_1 < n_2 < \cdots < n_i\} \subset \{1, \ldots, m-1\}$ is equal to $\binom{m-1}{i}$, we see that

$$P^u_{n+1,+}(\{\omega : \sharp(j : \varepsilon_j = -1) = i, L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\})$$

$$= \sum_{1 \le n_1 < n_2 < \cdots < n_i \le m-1} P^u_{n+1,+}(\{\omega : \{j : \varepsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\},$$

$$L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\})$$

$$= \binom{m-1}{i} u^{2m-2} x_u^{2m-1} (P^u_{n,+}(R_n \le k))^i, \quad 1 \le i \le m-1.$$

This is also true for $i = 0$.
Therefore, by summing up over i, we see that

$$P^u_{n+1,+}(\{\omega : L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\})$$

$$= u^{2m-2} x_u^{2m-1} (1 + P^u_{n,+}(R_n \le k))^{m-1}.$$

By summing up over m, we see that

$$P_{n+1,+}^u(R_{n+1} \le 2^{n+1} + k) = \sum_{m=1}^{\infty} u^{2m-2} x_u^{2m-1} (1 + P_{n,+}^u(R_n \le k))^{m-1}$$

$$= \Phi(A_{u,1}; P_{n,+}^u(R_n \le k)).$$

This completes the proof.   □

Next, we will show that $\tilde{g}_u$, which is the right continuous modification of g_u, satisfies (1.1) on $[0,1]$, not only on D. We define some notation. Let $X_n(x) = \lfloor 2^n x \rfloor - 2\lfloor 2^{n-1} x \rfloor$ and $\zeta_n(x) = \sum_{k=1}^{n} 2^{-k} X_k(x)$, $x \in [0,1)$, $n \ge 1$. Then, $\zeta_n(x) \le x < \zeta_n(x) + 2^{-n}$, $x \in [0,1)$, $n \ge 1$. Let $\gamma_u = 1/\Phi(A_{u,0}; 1)$. Let $p_{u,0}(z) = (z+1)/(z+\gamma_u)$ and $p_{u,1}(z) = 1 - p_{u,0}(z)$ for $z > -\gamma_u$. Let

$$\begin{pmatrix} p_{u,n}(x) & q_{u,n}(x) \\ r_{u,n}(x) & s_{u,n}(x) \end{pmatrix} = A_{u,X_1(x)} \cdots A_{u,X_n(x)}, \quad x \in [0,1), \ n \ge 1.$$

PROPOSITION 3.4.   (1) $g_u(\zeta_m(x)) = \Phi(A_{u,X_1(x)} \cdots A_{u,X_m(x)}; 0)$ and

$$g_u(\zeta_m(x) + 2^{-m}) = \Phi(A_{u,X_1(x)} \cdots A_{u,X_m(x)}; 1), \quad x \in [0,1), \ m \ge 1.$$

(2) $\tilde{g}_u = g_u$ on D.

(3) $\tilde{g}_u$ satisfies the equation (1.1) on $[0,1]$.

PROOF.   (1) Using (1.1), we can show the assertion by induction in n.

(2) By noting the definition of g_u and $\tilde{g}_u$, we have that $\tilde{g}_u(1) = 1 = g_u(1)$. Let $x \in D \cap [0,1)$. Then, there exists N such that $X_n(x) = 0$, $n > N$.

Then, by using the assertion (1),

$$\lim_{l \to \infty} g_u(x + 2^{-l}) = \lim_{l \to \infty} g_u(\zeta_l(x) + 2^{-l})$$

$$= \lim_{m \to \infty} \Phi(A_{u,X_1(x)} \cdots A_{u,X_N(x)}; \Phi(A_{u,0}^m; 1)).$$

Since $\Phi(A_{u,0}; \cdot)$ is a contraction map on $[0,1]$, $\lim_{m \to \infty} \Phi(A_{u,0}^m; 1) = 0$. Then, by using the assertion (1),

$$\lim_{m \to \infty} \Phi(A_{u,X_1(x)} \cdots A_{u,X_N(x)}; \Phi(A_{u,0}^m; 1)) = \Phi(A_{u,X_1(x)} \cdots A_{u,X_N(x)}; 0) = g_u(x).$$

Thus we obtain the assertion (2).

(3) Since $\tilde{g}_u(1) = 1$ and $\Phi(A_{u,1}; 1) = 1$, (1.1) holds for $x = 1$.

Let $x \in [0, 1/2)$. Then there exists a sequence $\{x_n\}_n \subset D \cap [0, 1/2)$ such that $x_n \downarrow x$. By using Proposition 3.3 and the assertion (2), $\tilde{g}_u(x_n) = \Phi(A_{u,0}; \tilde{g}_u(2x_n))$, $n \geq 1$. Since $\Phi(A_{u,0}; \cdot)$ is continuous and $\tilde{g}_u$ is right continuous, we have that $\tilde{g}_u(x) = \Phi(A_{u,0}; \tilde{g}_u(2x))$.

In the same manner, we see that $\tilde{g}_u(x) = \Phi(A_{u,1}; \tilde{g}_u(2x - 1))$ for $x \in [1/2, 1)$. Thus we obtain the assertion (3). □

PROOF OF THEOREM 1.2. First, we show the assertion (1). Let $\tilde{P}_n^u = P^u \circ ((R_n/2^n) - 1)^{-1}$. Let $\tilde{P}^u$ be the probability measure on $[0, 1]$ whose distribution function is $\tilde{g}_u$ and satisfying $\tilde{P}^u(\{0\}) = 0$. In other words, we will show that the function f_u in the statement in Theorem 1.2 is equal to $\tilde{g}_u$. It suffices to show that $\tilde{P}_n^u$ converges weakly to $\tilde{P}^u$, that is, for any continuous function f on $[0, 1]$,

$$\lim_{n \to \infty} \int_{[0,1]} f(x) \tilde{P}_n^u(dx) = \int_{[0,1]} f(x) \tilde{P}^u(dx). \tag{3.1}$$

Let $\varepsilon > 0$. Then, $\max_{1 \leq k \leq 2^m} |f(k/2^m) - f((k-1)/2^m)| < \varepsilon$ for some m. We have that

$$\left| \int_{[0,1]} f(x) \tilde{P}_n^u(dx) - \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \tilde{P}_n^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) \right| < \varepsilon, \tag{3.2}$$

and,

$$\left| \int_{[0,1]} f(x) \tilde{P}^u(dx) - \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \tilde{P}^u\left(\left(\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right) \right| < \varepsilon, \tag{3.3}$$

where we have used $\tilde{P}_n^u(\{1\}) = P^u(R_n = 2^{n+1}) = P_{n,+}^u(R_n = 2^{n+1}) = 0$ for the first inequality, and, $\tilde{P}^u(\{0\}) = 0$ for the second.

Let $n > m$. Then, by using Lemma 3.1, we see that for $1 \leq k \leq 2^m$,

$$\tilde{P}_n^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) = \tilde{P}_m^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) = g_u\left(\frac{k}{2^m}\right) - g_u\left(\frac{k-1}{2^m}\right).$$

By using Proposition 3.4(2), we see that for $1 \leq k \leq 2^m$,

$$\tilde{P}^u\left(\left(\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right) = \tilde{g}_u\left(\frac{k}{2^m}\right) - \tilde{g}_u\left(\frac{k-1}{2^m}\right) = g_u\left(\frac{k}{2^m}\right) - g_u\left(\frac{k-1}{2^m}\right).$$

Therefore, we see that

$$\sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \tilde{P}_n^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) = \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \tilde{P}^u\left(\left(\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right).$$

Recalling (3.2) and (3.3), we see that for any $n > m$,

$$\left| \int_{[0,1]} f(x) \tilde{P}_n^u(dx) - \int_{[0,1]} f(x) \tilde{P}^u(dx) \right| < 2\varepsilon.$$

Thus we see (3.1) and the proof of (1) completes.

The assertion (2) immediately follows from the definition of $\tilde{P}^u$ and Proposition 3.4(3).

Finally, we show the assertion (3). Let $u = 1$. Then, the absolute continuity of $\tilde{P}^1$ follows from [7], Theorem 1.2(1).

LEMMA 3.5. *Let $u \neq 1$. Let $x \in [0,1] \backslash D$. If $\tilde{g}_u$ is differentiable at x and $\tilde{g}_u'(x) \in [0, +\infty)$, then, $\tilde{g}_u'(x) = 0$.*

PROOF. We assume that there exists a point $x \in [0,1] \backslash D$ such that $\tilde{g}_u$ is differentiable at x and $\tilde{g}_u'(x) \in (0, +\infty)$.

Since $\tilde{g}_u$ is strictly increasing and $x \notin D$, we have that

$$\tilde{g}_u'(x) = \lim_{n \to \infty} 2^n(\tilde{g}_u(\zeta_n(x) + 2^{-n}) - \tilde{g}_u(\zeta_n(x))) = \lim_{n \to \infty} 2^n(g_u(\zeta_n(x) + 2^{-n}) - g_u(\zeta_n(x))).$$

Since $\tilde{g}_u'(x) \in (0, +\infty)$,

$$\lim_{n \to \infty} \frac{g_u(\zeta_{n+1}(x) + 2^{-(n+1)}) - g_u(\zeta_{n+1}(x))}{g_u(\zeta_n(x) + 2^{-n}) - g_u(\zeta_n(x))} = \frac{1}{2}.$$

Then, by using Proposition 3.4(1),

$$p_{u, X_{n+1}(x)}\left(\frac{r_{u,n}(x)}{s_{u,n}(x)}\right) = \frac{g_u(\zeta_{n+1}(x) + 2^{-(n+1)}) - g_u(\zeta_{n+1}(x))}{g_u(\zeta_n(x) + 2^{-n}) - g_u(\zeta_n(x))},$$

and, $\lim_{n \to \infty} p_{u, X_{n+1}(x)}(r_{u,n}(x)/s_{u,n}(x)) = 1/2$. Since $p_{u,1} = 1 - p_{u,0}$,

$$\lim_{n \to \infty} p_{u,i}(r_{u,n}(x)/s_{u,n}(x)) = 1/2 \quad \text{for } i = 0, 1.$$

Now we see that $\lim_{n \to \infty} r_{u,n}(x)/s_{u,n}(x) = \gamma_u - 2$. Since $x \notin D$, there exists infinitely many natural numbers n such that $X_n(x) = i$ for each $i = 0, 1$. Since $r_{u,n+1}(x)/s_{u,n+1}(x) = \Phi({}^t A_{u, X_{n+1}(x)}; r_{u,n}(x)/s_{u,n}(x))$, we see that $\Phi({}^t A_{u,i}; \gamma_u - 2) =$

$\gamma_u - 2$ for each $i = 0, 1$. This is true if and only if $u = 1$. But this contradicts the assumption. $\qquad\square$

Let $u \neq 1$. Then, by noting Lemma 3.5 and the Lebesgue differentiation theorem, we see that $\tilde{g}'_u = 0$ a.e. and $\tilde{P}^u$ is singular. These complete the proof of (3). $\qquad\square$

PROOF OF THEOREM 1.4.   In this proof, we write $\Phi_{u,i}(z) = \Phi(A_{u,i}; z)$, $i = 0, 1$. We first explain the meaning of the value $u = \sqrt{3}$. By explicit calculation, we see that if $u < \sqrt{3}$, then, $0 < \Phi'_{u,1}(z) < 1$, $z \in [0, 1]$, namely, $\Phi_{u,1}(\cdot)$ is a contraction map on $[0, 1]$, and de Rham's theory [3] is applicable to $(A_{u,0}, A_{u,1})$ in the form of [7]. In contrast, this property fails if $u \geq \sqrt{3}$. In fact, $\Phi'_{\sqrt{3},1}(z) \leq 1$, with $\Phi'_{\sqrt{3},1}(z) = 1$ implying $z = 1$. If $u > \sqrt{3}$, there exists $z_0 = z_0(u) \in (0, 1)$ such that $\Phi'_{u,1}(z) < 1$ for $z < z_0$, and $\Phi'_{u,1}(z) > 1$ for $z > z_0$.

We now turn to the proof of the theorem. We denote $f^{m+1} = f \circ f^m$, $m \geq 1$, for $f : [0, 1] \to [0, 1]$.

(1) If $0 < u < \sqrt{3}$, then, $(A_{u,0}, A_{u,1})$ satisfies the conditions (A1)–(A3) in [7] and hence $\tilde{P}^u$ has no atoms.

Let $u = \sqrt{3}$. Let $h_i = \Phi_{\sqrt{3},i}$, $i = 0, 1$. Then we have the following results by computations.

LEMMA 3.6.   (1) $h_0(z) < h_1(z)$ for $z \in [0, 1]$.
(2) h'_i, $i = 0, 1$, are strictly increasing on $(0, 1)$.
(3) $h'_0(z) \leq 3h'_1(z)$ for $z \in (0, 1)$.
(4) $h'_0(z) \leq h'_1(z)$ for $z \geq h_1^2(0)$.

Now it is sufficient to show the following.

$$\lim_{m \to \infty} \max_{1 \leq k \leq 2^m} \left\{ g_{\sqrt{3}}\left(\frac{k}{2^m}\right) - g_{\sqrt{3}}\left(\frac{k-1}{2^m}\right) \right\} = 0. \tag{3.4}$$

Let $m \geq 3$ and $1 \leq k \leq 2^m$. Let $x_i = X_i((k-1)/2^m)$, $1 \leq i \leq m$. Then,

$$g_{\sqrt{3}}\left(\frac{k}{2^m}\right) - g_{\sqrt{3}}\left(\frac{k-1}{2^m}\right) = h_{x_1} \circ \cdots \circ h_{x_m}(1) - h_{x_1} \circ \cdots \circ h_{x_m}(0)$$

$$= \int_0^1 (h_{x_1} \circ \cdots \circ h_{x_m})'(x)\, dx$$

$$= \int_0^1 h'_{x_1}(h_{x_2} \circ \cdots \circ h_{x_m}(x)) \cdots h'_{x_{m-1}}(h_{x_m}(x))h'_{x_m}(x)\, dx$$

$$\leq \int_0^1 h'_{x_1}(h_1^{m-1}(x)) \cdots h'_{x_{m-1}}(h_1(x)) h'_{x_m}(x)\, dx$$

$$\leq \int_0^1 h'_1(h_1^{m-1}(x)) \cdots 3h'_1(h_1(x)) 3h'_1(x)\, dx$$

$$= 9 \int_0^1 (h_1^m)'(x)\, dx = 9(1 - h_1^m(0)),$$

where we have used Proposition 3.4 (1) for the first equality, Lemma 3.6 (1) and (2) for the fourth inequality, and, Lemma 3.6 (3) and (4) for the fifth. Since $h_1^n(0) = n/(n+1)$, $n \geq 1$, we see that $\lim_{n \to \infty} h_1^n(0) = 1$. Thus we see (3.4) and the proof of the assertion (1) completes.

(2) Let $x \in D \cap (0, 1)$. Let $x_i = X_i(x)$, $i \geq 1$. Then, there exists a unique $m \geq 1$ such that $x_m = 1$ and $x_i = 0$, $i \geq m + 1$. Let $\phi = \Phi_{u, x_1} \circ \cdots \circ \Phi_{u, x_{m-1}} \circ \Phi_{u, 0}$. Let $n > m$ and $y_i = X_i(x - (1/2^n))$. Then, we have that $y_i = x_i$, $1 \leq i \leq m - 1$, $y_m = 0$, $y_i = 1$, $m + 1 \leq i \leq n$, and, $y_i = 0$, $i > n$. By noting Proposition 3.4 (1) and $\Phi_{u, 0}(1) = \Phi_{u, 1}(0)$, we have that

$$g_u(x) = \phi(1), \quad g_u\left(x - \frac{1}{2^n}\right) = \phi(\Phi_{u, 1}^{n-m}(0)). \tag{3.5}$$

Note that $\Phi_{u, 1}$ is increasing and strictly convex, $\Phi_{u, 1}(0) > 0$, $\Phi_{u, 1}(1) = 1$, and, $\Phi'_{u, 1}(1) > 1$. Therefore, there exists $z_1 \in (0, 1)$ such that

$$\Phi_{u, 1}(z_1) = z_1, \quad \Phi_{u, 1}(z) > z, \quad z \in (0, z_1), \quad \Phi_{u, 1}(z) < z, \quad z \in (z_1, 1).$$

Then, $z_1 = \lim_{n \to \infty} \Phi_{u, 1}^n(0)$ and $\Phi_{u, 1}^n(0) \leq z_1 < 1$, $n \geq 1$.

We have that for $n > m$,

$$\tilde{P}^u\left(\left(x - \frac{1}{2^n}, x\right]\right) = g_u(x) - g_u\left(x - \frac{1}{2^n}\right)$$

$$= \phi(1) - \phi(\Phi_{u, 1}^{n-m}(0))$$

$$\geq \phi(1) - \phi(z_1),$$

where we have used Proposition 3.4 (2) for the first equality, and, (3.5) for the second. Letting $n \to \infty$, we have that $\tilde{P}^u(\{x\}) \geq \phi(1) - \phi(z_1) > 0$.

We can show that $\tilde{P}^u(\{1\}) > 0$ in the same manner. These complete the proof of the assertion (2). □

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