A Class of Hypoelliptic Vishik-Wentzell Boundary Value Problems

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Subtitle

Spectral Analysis
of
Vishik-Wentzell Boundary Value Problems

Purpose

The purpose of my talk is to study Vishik-Wentzell Boundary Value Problems for second-order, elliptic differential operators in the framework of Sobolev spaces.

References

Visik, M.I.: On general boundary value problems for elliptic differential equations.

English translation: American Mathematical translation (2) 24 (1963), 107-172.

Hormander, L.: Linear partial differential operators. Springer-Verlag, 1963.

Mark Iosifovich Vishik

M. I. Vishik (1921-2011)
Soviet Mathematician

Lars Hormander

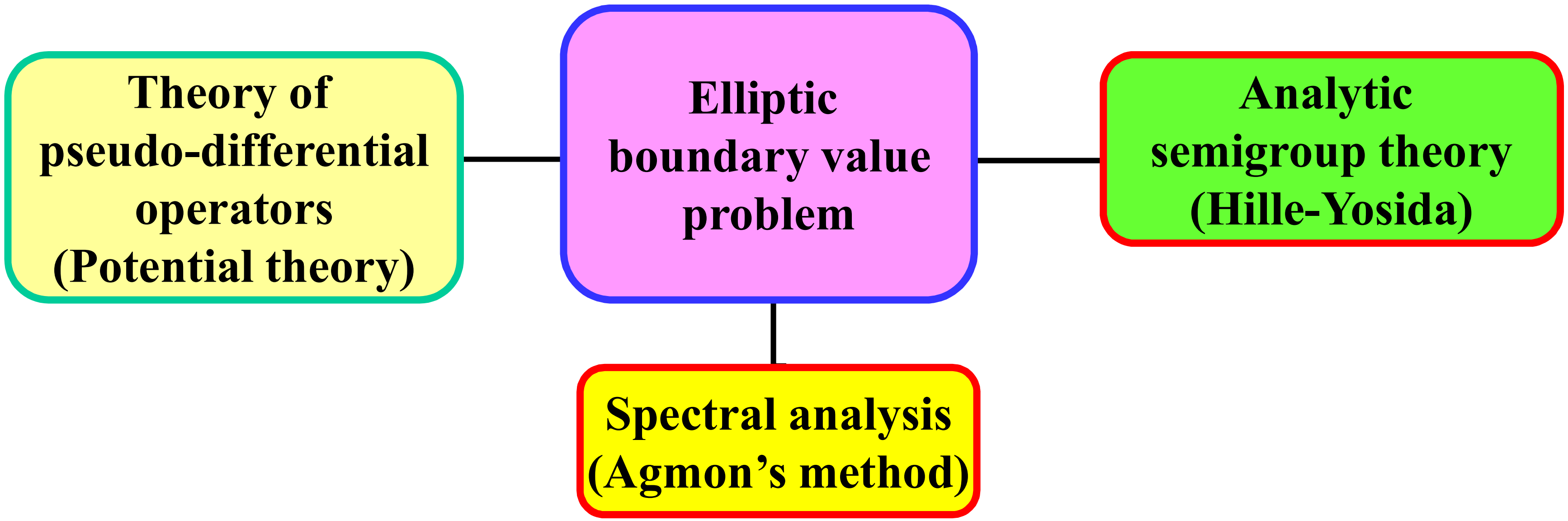
Lars Hormander (1931-2012)
Swedish Mathematician

Alexander Dmitrievich Wentzell

A. D. Wentzell (1937-)
Soviet-American Mathematician

Bird's-Eye View

Functional analytic approach to elliptic boundary value problems



References

References 1 (My Work)

- **K. Taira: Un theoreme d'existence et d'unicite des solutions pour des problemes aux limites non-elliptiques. J. Functional Analysis, 43 (1981), 166-192.**

  DOI: 10.1016/0022-1236(81)90027-6

References 2 (My Work)

- **Taira**: **Analytic Semigroups and Semilinear Initial Boundary Value Problems**, London Mathematical Society Lecture Note Series, No. 434, Cambridge University Press (2016)

  ISBN: 978-1-316-62086-1
- **Taira**: **Spectral Analysis of the Subelliptic Oblique Derivative Problem**, Arkiv for Matematik 55 (2017), 243-270.

  DOI: 10.4310/ARKIV.2017.v55.n1.a13

References 3 (My Work)

- **Taira**: **Spectral Analysis of the Hypoelliptic Robin Problem**, **Annali dell'Universit`a di Ferrara, 65 (2019), 171-199.**

  DOI: 10.1007/s11565-018-0308-4

Concrete Example

Concrete Example ($n=3$)

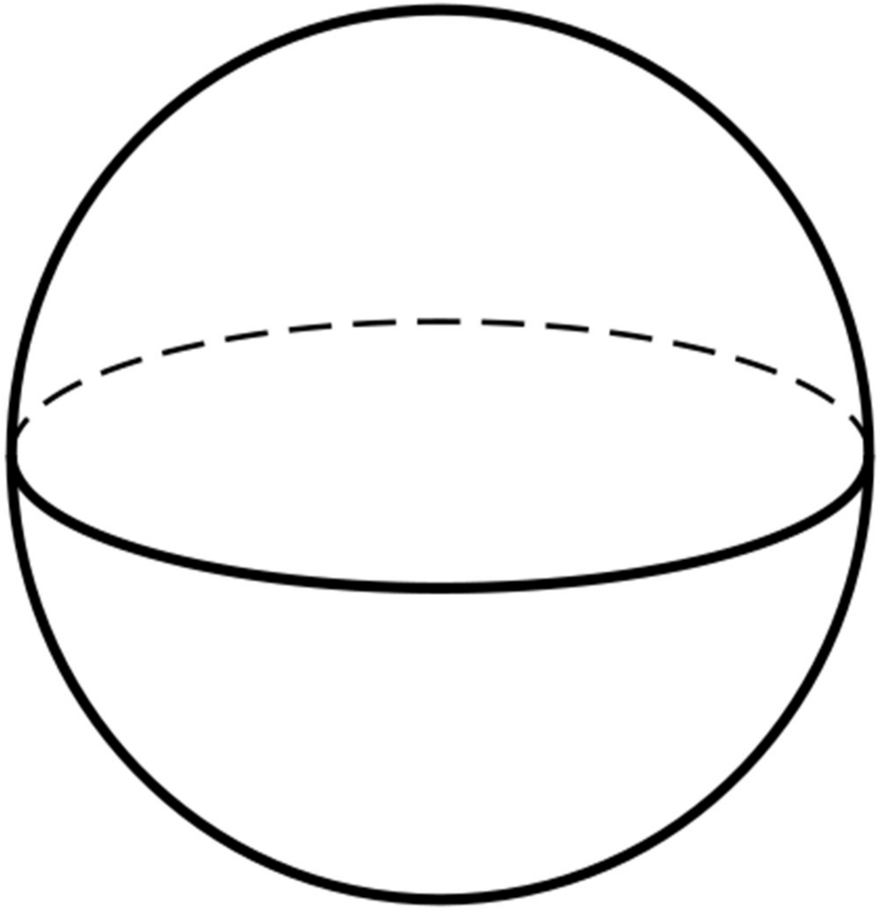
$\Omega = \left\{ x^2 + y^2 + z^2 \le 1 \right\} = \left\{ r \le 1 \right\}$ (**unit ball**)

$\Gamma = \left\{ x^2 + y^2 + z^2 = 1 \right\} = \left\{ r = 1 \right\}$ (**unit sphere**)

$$\begin{cases} x = r\cos\theta\cos\omega \\ y = r\cos\theta\sin\omega \\ z = r\sin\theta \end{cases}$$

$$\begin{pmatrix} 0 \le r \le 1 \\ -\pi/2 \le \theta \le \pi/2 \\ 0 \le \omega \le 2\pi \end{pmatrix}$$

Unit Ball



$\theta = 0$

(Equator)

Vishik-Wentzell Boundary Value Problem

$$\Delta u = f \quad \text{in } \Omega = \{r \le 1\},$$

$$\Lambda u = -\frac{\partial u}{\partial r} + \left(\frac{\partial^2 u}{\partial \theta^2} + \theta^2 \frac{\partial^2 u}{\partial \omega^2} + \left(\theta^2 - \frac{\pi^2}{4} \right) u \right)$$

$$= 0 \quad \text{on } \Gamma = \{r = 1\}$$

$$\mu(x') \equiv 1$$

Commutators (1)

$$X = \frac{\partial}{\partial \theta}, \quad Y = \theta \frac{\partial}{\partial \omega}$$

$$[X, Y] = XY - YX = \frac{\partial}{\partial \omega}$$

$$\Rightarrow$$

$$\text{span}\{X, Y, [X, Y]\} = \mathbf{R}^2$$

Commutators (2)

$$X = \frac{\partial}{\partial\theta}, \quad Y = \theta^2 \frac{\partial}{\partial\omega}$$

$$[X, Y] = XY - YX = 2\theta \frac{\partial}{\partial\omega}$$

$$[X, [X, Y]] = 2\frac{\partial}{\partial\omega}$$

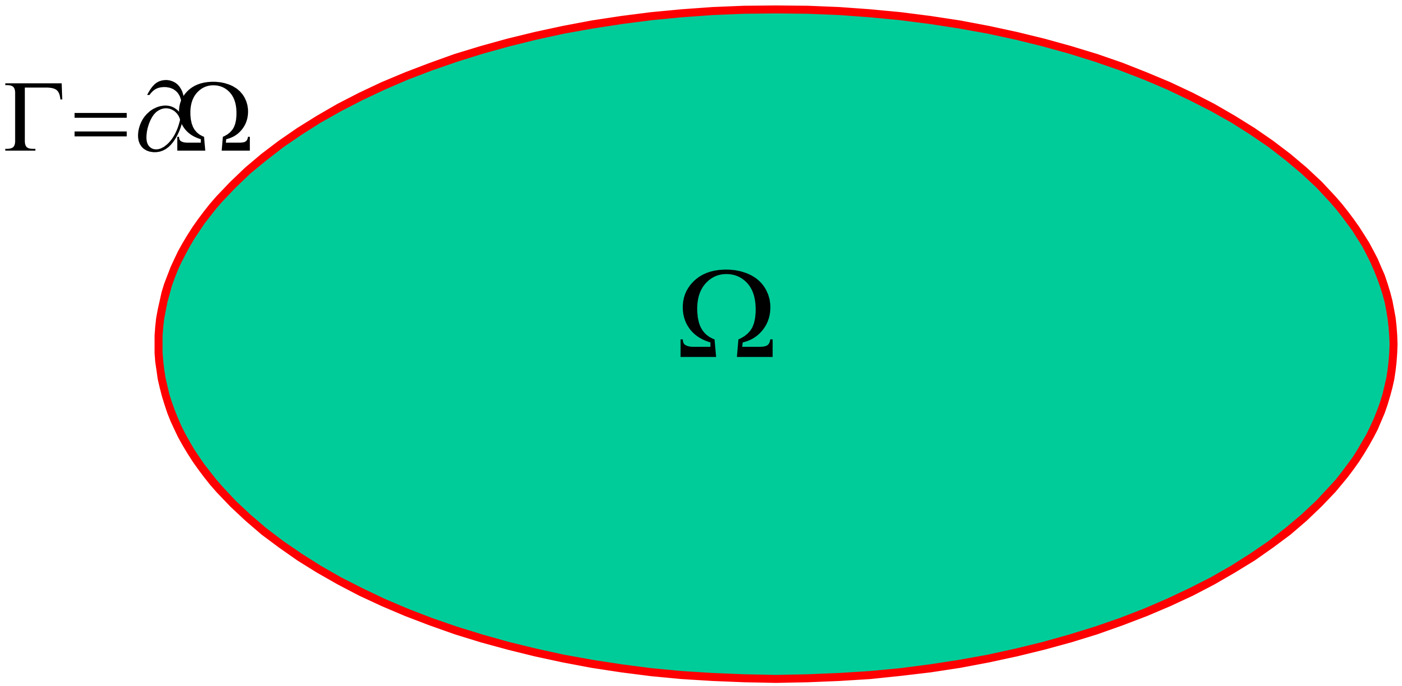
$$\Rightarrow$$

$$\boxed{\operatorname{span}\{X, Y, [X, Y], [X, [X, Y]]\} = \mathbf{R}^2}$$

Formulation of the Problem

Bounded Domain

$\mathbf{R}^n, \quad n \geq 2$



Laplace Operator

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

Vishik-Wentzell Boundary Condition

$$\Lambda u(x') = \mu(x') \frac{\partial u}{\partial \boldsymbol{n}} + Qu = 0 \text{ on } \Gamma.$$

(1) $\mu(x') \in C^{\infty}(\Gamma)$ and $\mu(x') \geq 0$ on Γ.

(2) $$Qu = \sum_{i,j=1}^{n-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} \beta^i(x') \frac{\partial u}{\partial x_i} + \gamma(x') u$$

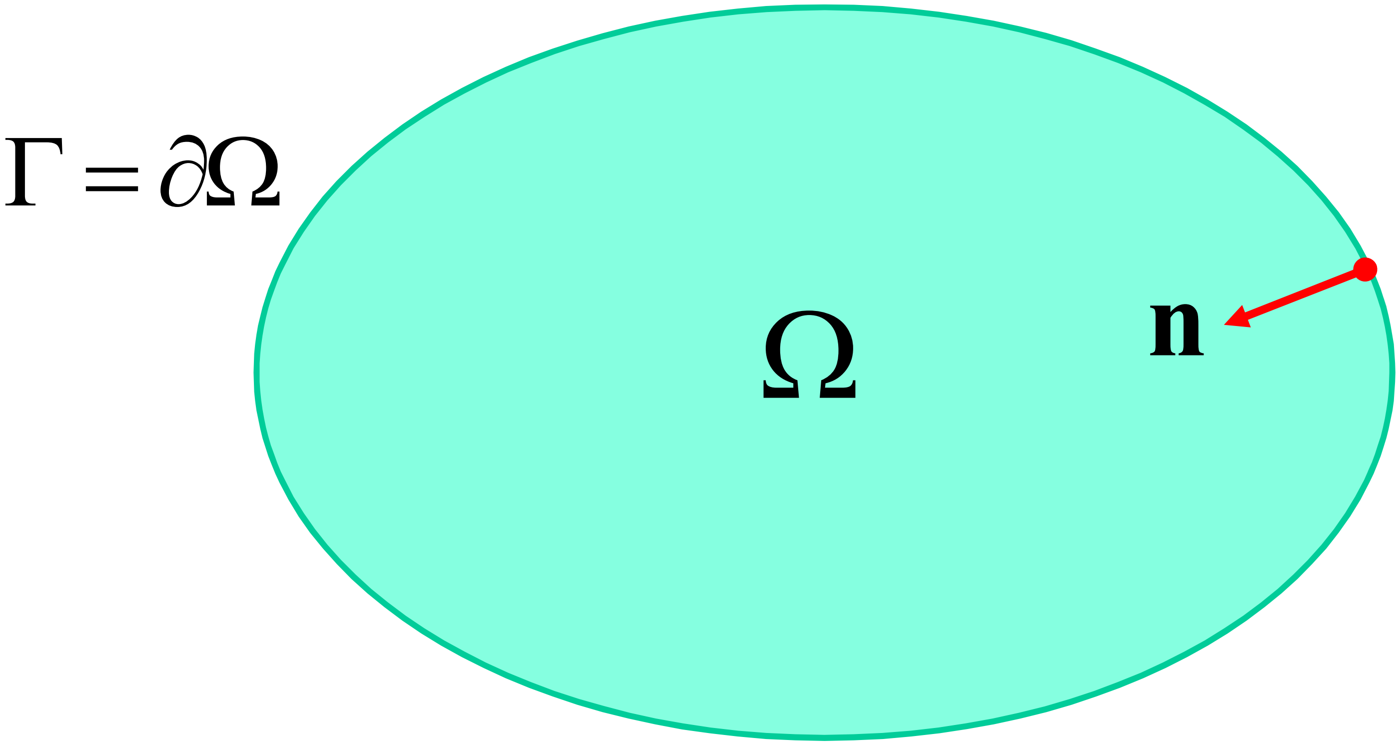
(3) $Q1(x') = \gamma(x') \leq 0$ on Γ.

Normal Derivative

$$\frac{\partial}{\partial \boldsymbol{n}} = \sum_{i=1}^{n} n_i \frac{\partial}{\partial x_i}$$

$\mathbf{n} = (n_1, n_2, \cdots, n_n)$ is the unit **interior** **normal**.

Normal Derivative



Order of the Vishik Boundary Condition

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

The order of $\Delta = 2$

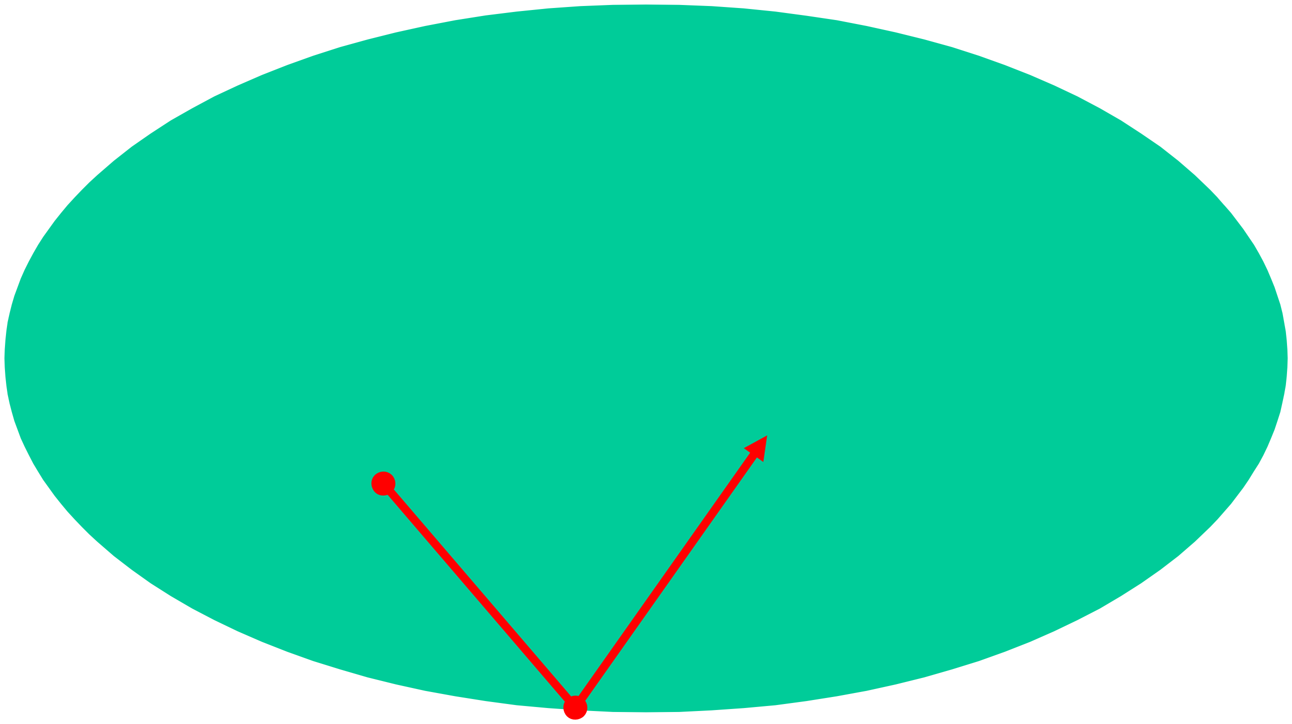
$$\Lambda u(x') = \mu(x') \frac{\partial u}{\partial \boldsymbol{n}} + Qu = 0 \text{ on } \Gamma.$$

The order of $Q > 1 = 2 - 1$

Probabilistic Meaning
of
Three Terms

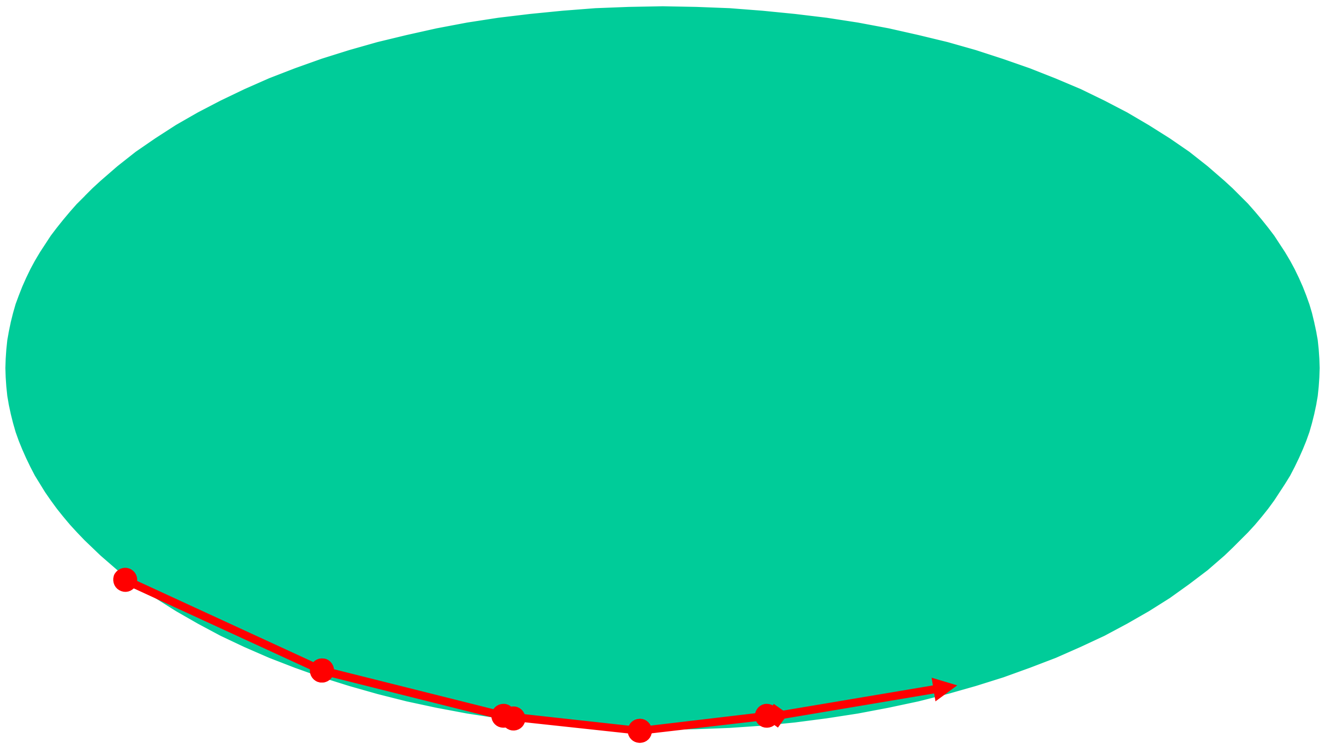
Reflection Phenomenon

(Neumann Condition)



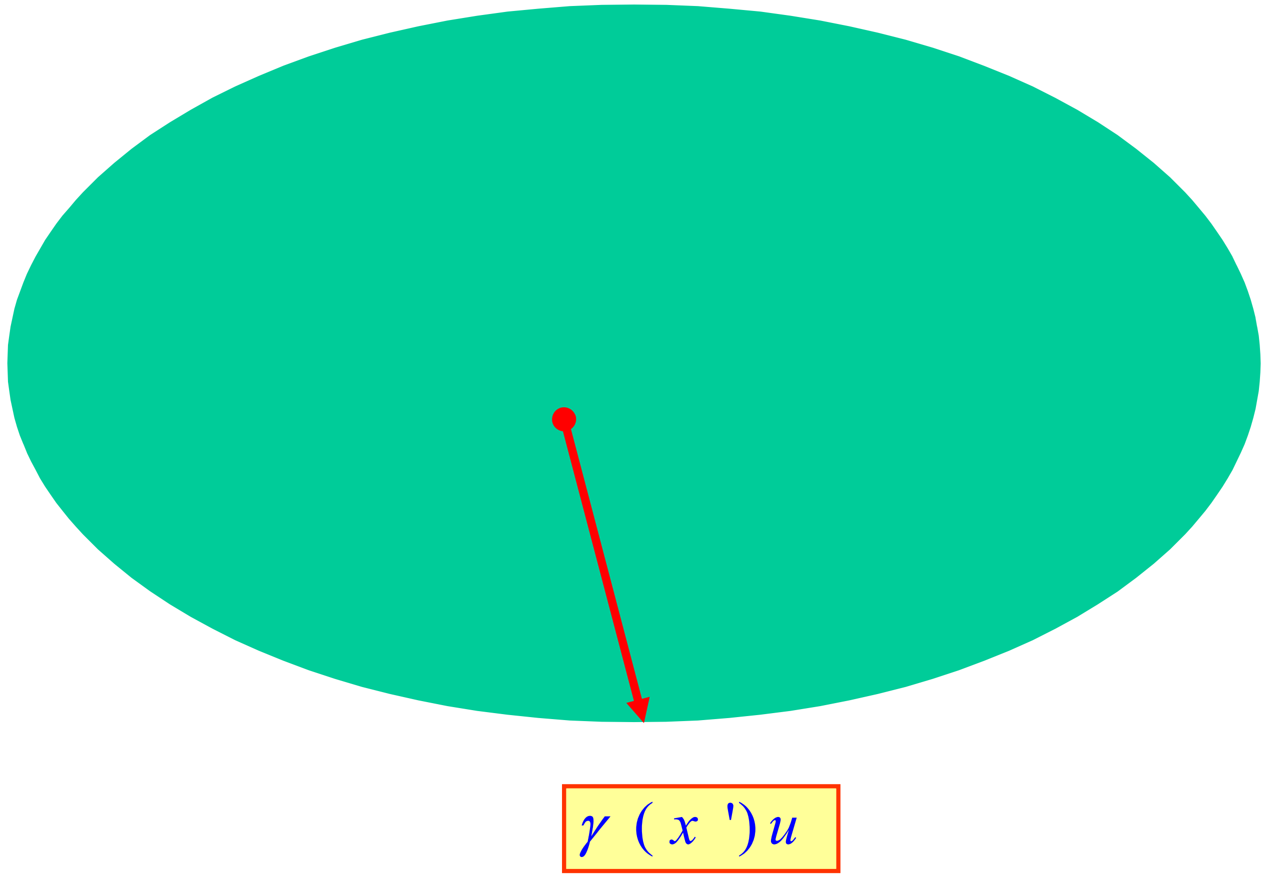
$$\mu(x')\frac{\partial u}{\partial \boldsymbol{n}}$$

Diffusion Phenomenon along the Boundary
(degenerate diffusion process on the boundary)



$Qu(x')$

Absorption Phenomenon
(Dirichlet Condition)



Main Results

Fundamental Hypothesis

$$\Lambda u(x') = \mu(x') \frac{\partial u}{\partial \boldsymbol{n}} + Qu = 0 \text{ on } \Gamma.$$

$$\text{(H)} \quad \boxed{\mu(x') > 0 \text{ on } \Gamma}$$

L^2 Approach

Existence and Uniqueness Theorem

The linear **homogeneous** problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \Lambda u = 0 & \text{on } \Gamma \end{cases}$$

has a **unique** solution $u \in H^{s+2}(\Omega)$

for **any** $f \in H^s(\Omega)$.

Here $\forall s \geq 0$.

The **elliptic gain** of 2-derivatives just as in the **Neumann case**

Hille-Yosida Theory of Semigroups

Closed Realization

We define a linear operator

$$\mathfrak{A}_2 : L^2(\Omega) \to L^2(\Omega)$$

as follows:

(a) The domain $D(\mathfrak{A}_2)$ is the set

$$D(\mathfrak{A}_2) = \{u \in H^2(\Omega) : \boxed{\Lambda u = 0 \text{ on } \Gamma}\}.$$

(b) $\mathfrak{A}_2 u = \Delta u, \ \forall u \in D(\mathfrak{A}_2).$

$\Longrightarrow$

$\mathfrak{A}_2$ is a **densely defined, closed operator**

Sharp Resolvent Estimates

$$\left\| \left(\mathfrak{A}_2 - \lambda I \right)^{-1} f \right\|_{L^2(\Omega)} \leq \frac{C}{|\lambda|} \| f \|_{L^2(\Omega)}$$

$$\forall f \in L^2(\Omega)$$

Generation Theorem of an Analytic Semigroup

We define a **densely defined, closed** operator

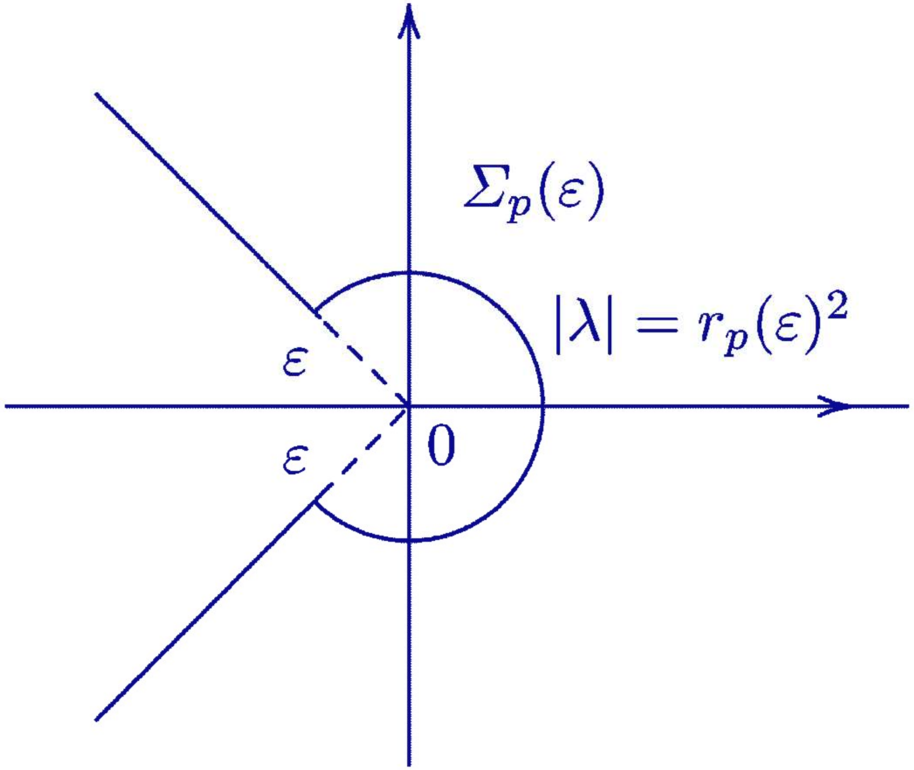
$\mathfrak{A}_2 : L^2(\Omega) \to L^2(\Omega)$

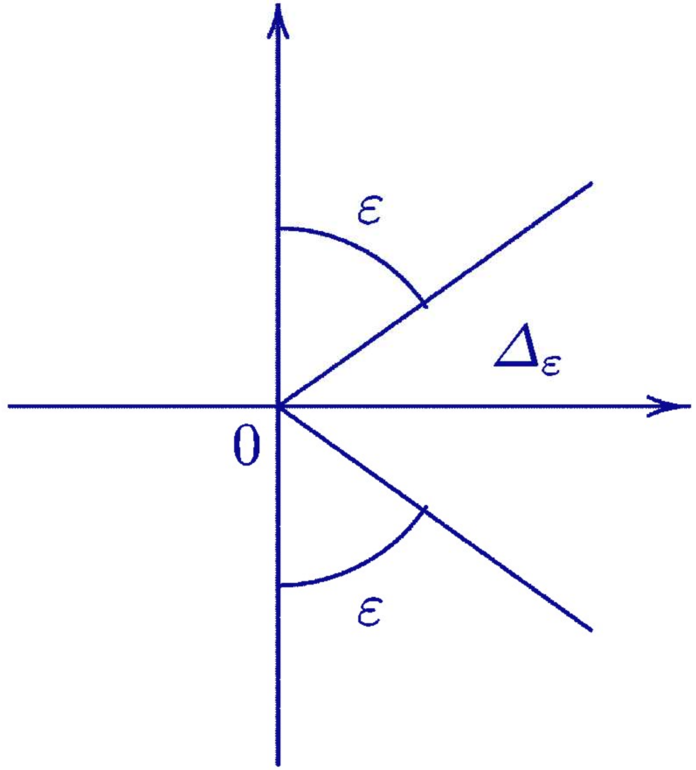
as follows:

(a) $D(\mathfrak{A}_2) = \left\{ u \in H^2(\Omega) : \boxed{\Lambda u = 0 \text{ on } \Gamma} \right\}$

(b) $\mathfrak{A}_2 u = \Delta u, \ \forall u \in D(\mathfrak{A}_2)$

Then $\mathfrak{A}_2$ generates an **analytic semigroup** $e^{z\mathfrak{A}_2}$ on $L^2(\Omega)$





$$\lambda = r^2 e^{i\theta}$$

$$-\pi + \varepsilon \le \theta \le \pi - \varepsilon$$

$$z \in \Delta_\varepsilon$$

Eigenvalue Distribution

Let

$$N(t) := \sum_{\mathrm{Re}\,\lambda_j \geq -t} 1$$ (the counting function)

where each λ_j is repeated according to its multiplicity.

Then the **asymptotic eigenvalue distribution** formula

$$N(t) = \frac{|\Omega|}{2^n \pi^{n/2} \Gamma(n/2+1)} \cdot t^{n/2} + o(t^{n/2}) \text{ as } t \to +\infty$$

holds true.

Concrete Example

Typical Example ($n=3$)

$$\Omega=\left\{x^2+y^2+z^2\le 1\right\}=\left\{r\le 1\right\}$$ (**unit ball**)

$$\Gamma=\left\{x^2+y^2+z^2=1\right\}=\left\{r=1\right\}$$ (**unit sphere**)

$$\begin{cases} x=r\cos\theta\cos\omega \\ y=r\cos\theta\sin\omega \\ z=r\sin\theta \end{cases}$$

$$\begin{pmatrix} 0\le r\le 1 \\ -\pi/2\le\theta\le\pi/2 \\ 0\le\omega\le 2\pi \end{pmatrix}$$

Vishik-Wentzell Eigenvalue Problem

$$\Delta u = \lambda u \quad \text{in } \Omega = \{r \le 1\},$$

$$\Lambda u = -\frac{\partial u}{\partial r} + \left(\frac{\partial^2 u}{\partial \theta^2} + \exp\left[-\frac{2}{\theta^2} \right] \frac{\partial u}{\partial \omega} + \left(\theta^2 - \frac{\pi^2}{4} \right) u \right)$$

$$= 0 \quad \text{on } \Gamma = \{r = 1\}$$

$$\mu(x') \equiv 1$$

Asymptotic Eigenvalue Distribution

Let

$$N(t) := \sum_{\mathrm{Re}\,\lambda_j \geq -t} 1,$$

where each λ_j is repeated according to its multiplicity.

Then the **asymptotic eigenvalue distribution** formula

$$\boxed{N(t) = \frac{2}{9\pi}\, t^{3/2} + o(t^{3/2}) \ \text{ as } t \to +\infty}$$

Crucial Points

Two Crucial Points

(1) How to define the **Vishik-Wentzell Boundary Conditions** in the framework of **Sobolev spaces.**

(2) How to study the **Fredholm boundary pseudo-differential operator**, by using the theory of pseudo-differential operators.

Definition
of
Vishik-Wentzell Boundary Conditions

A Modern Version
of
the Classical Potential Theory

Alberto Pedro Calderon

Ealberto Pedro Calderon (1920-1998)
Algentinian Mathematician

Robert T. Seeley

Robert T. Seeley (1932-2016)
American Mathematician

References

- **Seeley**: **Singular Integrals and Boundary Value Problems**, **American Journal of Mathematics, 88 (1966), 781-809.**

Maximal Domain

We define the **maximal domain**

$$H_{\Delta}(\Omega) = \left\{ u \in L^2(\Omega) : \Delta u \in L^2(\Omega) \right\}$$

with the **graph norm**

$$\|u\|_{H_{\Delta}(\Omega)} = \sqrt{\|u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2}$$

Definition of the boundary condition (1)

$$\Lambda u = \mu(x')\frac{\partial u}{\partial \boldsymbol{n}} + Q\left(u\big|_{\Gamma}\right) \in H^{-5/2}(\Gamma)$$

$$\forall u \in H_{\Delta}(\Omega)$$

Definition of the boundary condition (2)

$v = \exists! G_D f \in H^2(\Omega)$ such that

$$\begin{cases} \Delta v = f \text{ in } \Omega, \\ v = 0 \quad \text{on } \Gamma \end{cases}$$

(**Dirichlet problem**)

Definition of the boundary condition (3)

(1) $G_D : H^s(\Omega) \to H^{s+2}(\Omega)$

(2) $\Lambda(G_D f) = \mu(x')\dfrac{\partial}{\partial \boldsymbol{n}}(G_D f) + Q\left(G_D f\big|_{\Gamma}\right)$

$$= \mu(x')\frac{\partial}{\partial \boldsymbol{n}}(G_D f) + Q(0)$$

$$= \mu(x')\frac{\partial}{\partial \boldsymbol{n}}(G_D f) \in H^{s+1/2}(\Gamma)$$

gain of (1/2 - (-1/2) =) one derivative

Definition of the boundary condition (4)

$$w := u - v = u - G_D f \in L^2(\Omega)$$

$$\boxed{\Delta w = 0 \text{ in } \Omega}$$

$$\Leftrightarrow$$

$$\boxed{w = P\psi, \quad \exists \psi \in H^{-1/2}(\Gamma)}$$

(Poisson kernel)

Definition of the boundary condition (5)

$$u \in H_{\Delta}(\Omega)$$

$$\Leftrightarrow$$

$$u = G_D(\Delta u) + P\psi,$$

$$\psi \in H^{-1/2}(\Gamma)$$

Definition of the boundary condition (6)

$$u \in H_{\Delta}(\Omega)$$

$$\Leftrightarrow$$

$$\boxed{\Lambda u := \Lambda\Big(G_D\big(\Delta u\big)\Big) + \Lambda\big(P\psi\big) \in H^{-5/2}(\Gamma)}$$

$$\Lambda\Big(G_D\big(\Delta u\big)\Big) = \mu(x')\frac{\partial}{\partial \boldsymbol{n}}\Big(G_D\big(\Delta u\big)\Big) \in H^{1/2}(\Gamma)$$

$$\Lambda\left(P\psi\right) = Q\psi + \mu(x')\frac{\partial}{\partial \boldsymbol{n}}\left(P\psi\right)$$

$$= Q\psi + \mu(x')\Pi\,\psi \in H^{-5/2}(\Gamma)$$

Fredholm
Boundary Operator

Erik Ivar Fredholm

Erik Ivar Fredholm (1866-1927)
Swedish Mathematician

Fredholm Boundary Operator

$$T\psi = (\Lambda P)\psi = Q\psi + \mu(x')\frac{\partial}{\partial \boldsymbol{n}}(P\psi)$$

$$= \sum_{i,j=1}^{n-1} \alpha^{ij}(x')\frac{\partial^2 \psi}{\partial x_i \partial x_j} + \sum_{j=1}^{n-1} \beta^j(x')\frac{\partial \psi}{\partial x_j}$$

$$+\gamma(x')\psi + \mu(x')\frac{\partial}{\partial \boldsymbol{n}}(P\psi)$$

Dirichlet-to-Neumann Operator

$$\Pi\psi = \frac{\partial}{\partial \boldsymbol{n}}\left(P\psi\right)\Big|_{\Gamma}, \quad \forall\psi \in C^{\infty}(\Gamma)$$

$$\Pi = \sqrt{-\Delta'} = \left(-\Delta'\right)^{1/2} \in L^{1}_{\mathrm{cl}}(\Gamma),$$

Δ' = **Laplace - Beltrami operator on** Γ

Fredholm Alternative

Fundamental Hypothesis

(H) $\mu(x') > 0$ on Γ

Index Formula

We define a **densely defined, closed** operator

$\mathfrak{A}_2 : L^2(\Omega) \to L^2(\Omega)$

as follows:

(a) $\boxed{D(\mathfrak{A}_2) = \{u \in H^2(\Omega) : \Lambda u = 0 \text{ on } \Gamma\}}$

(b) $\mathfrak{A}_2 u = \Delta u, \ \forall u \in D(\mathfrak{A}_2)$

Then $\mathfrak{A}_2 - \lambda I$ is a **Fredholm operator**

with **index zero** for $\forall \lambda \in \mathbf{C}$.

Fredholm Alternative

$$\mathfrak{A}_2 \text{ injective} \Leftrightarrow \mathfrak{A}_2 \text{ surjective}$$

Regularity Theorem

Sobolev Regularity Theorem

Then we have, for all $s \geq 0$,

$$\begin{cases} u \in L^2(\Omega), \\ \Delta u = f \in H^s(\Omega), \\ \qquad \Lambda u = 0 \end{cases}$$

$$\Rightarrow u \in H^{s+2}(\Omega).$$

The elliptic gain of 2-derivatives as in the **Neumann case**

Regularity Theorem (1)

$$u \in H_{\Delta}(\Omega) \Leftrightarrow \begin{cases} u \in L^2(\Omega), \\ \Delta u \in L^2(\Omega), \\ \Lambda u = 0 \end{cases}$$

$$\Rightarrow u \in H^2(\Omega)$$

Regularity Theorem (2)

$$\begin{cases} u \in L^2(\Omega), \\ \Delta u = 0, \\ \Lambda u = 0 \end{cases}$$

$$\Rightarrow u \in C^{\infty}\left(\overline{\Omega}\right)$$

Uniqueness Theorem

Uniqueness Theorem

If a function

$$u \in L^2(\Omega)$$

is a solution of the homogeneous problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \Lambda u = 0 & \text{on } \Gamma, \end{cases}$$

then it follows that

$$u = 0 \text{ in } \Omega.$$

Sobolev Regularity Theorem

$+$

Classical Maximum Principle

$\Rightarrow$

Uniqueness Theorem

Maximum Principle
(Sobolev Space Version)

Maximum Principle
(Aleksandrov-Bakel'man)

Assume that:

$$u \in C(\overline{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega),$$

$$Au(x) \leq 0 \quad \text{for a.a. } x \in \Omega$$

Then:

$$\sup_{x \in \Omega} u(x) \leq \sup_{x' \in \Gamma} \max\{u(x'), 0\}.$$

Strong Maximum Principle

Assume that

$$u \in C(\overline{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega),$$

$$Au(x) \le 0 \quad \text{for a.a. } x \in \Omega,$$

$$m = \sup_{\Omega} u \ge 0.$$

Then:

$$\exists x_0 \in \Omega \text{ s.t. } u(x_0) = m \Rightarrow u(x) \equiv m, \quad \forall x \in \Omega.$$

Hopf Boundary Point Lemma

Assume that

$$\begin{cases} u \in C^1(\overline{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega), \\ Au(x) \le 0 \quad \text{for a.a. } x \in \Omega. \end{cases}$$

$\exists x_0' \in \Gamma$ such that $u(x_0') = \sup\limits_{\Omega} u = m \ge 0$

$u(y) < m, \ \forall y \in \Omega$

Then :

$$\frac{\partial u}{\partial \mathbf{n}}(x_0') > 0.$$

Fredholm Alternative

$$\mathfrak{A}_2 \text{ injective} \Leftrightarrow \mathfrak{A}_2 \text{ surjective}$$

Existence and Uniqueness Theorem

Existence and Uniqueness Theorem

The linear **homogenous** problem

$$\begin{cases} \Delta u = f \quad \text{in } \Omega, \\ \Lambda u = 0 \quad \text{on } \Gamma \end{cases}$$

has a **unique** solution $u \in H^{s+2}(\Omega)$

for **any** $f \in H^{s}(\Omega)$.

The **elliptic gain** of 2-derivatives

as in the **Neumann case**

Uniqueness Theorem

$$+$$

Fredholm Alternative

$$\Rightarrow$$

Existence Theorem

Spectral Analysis

Regularity of the Resolvent

$$\begin{cases} \Delta u = f \in H^s(\Omega) \\ \quad \Lambda u = 0 \ \text{ on } \Gamma \end{cases}$$

$$\Rightarrow$$

$$u = \mathfrak{A}_2^{-1} f \in H^{s+2}(\Omega)$$

Elliptic gain of 2-derivatives from f

Conclusions (Agmon's theory)

(1) The spectrum of $\mathfrak{A}_2$ is **discrete** and the eigenvalues λ_j of $\mathfrak{A}_2$ have finite multiplicities.

(2) All rays different from the **negative axis** are rays of minimal growth of the resolvent $\left(\mathfrak{A}_2 - \lambda I\right)^{-1}$.

(3) The negative axis is a **direction of condensation** of eigenvalues λ_j of $\mathfrak{A}_2$.

Idea of Proof

Special Reduction to the Boundary

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \Lambda u = 0 & \text{on } \Gamma. \end{cases}$$

$$\Updownarrow$$

$$T\psi = Q\psi + \mu(x')\frac{\partial}{\partial \boldsymbol{n}}\left(P\psi\right)$$

$$= -\mu(x')\frac{\partial}{\partial \boldsymbol{n}}\left(G_D f\right)$$

Reduction to the Boundary

$$\mu(x') > 0 \quad \text{on } \Gamma$$

$$u = G_D f + P\left(\exists \boldsymbol{T}^{-1} \left(\mu(x') \frac{\partial}{\partial \boldsymbol{n}} \left(G_D f \right) \right) \right) \text{ on } \Gamma$$

Energy Estimates

Fredholm Boundary Operator

$$T\psi = \mu(x')\Pi\psi + Q\psi$$

$$= \mu(x')\Pi\psi$$

$$+ \sum_{i,j=1}^{n-1} \alpha^{ij}(x')\frac{\partial^2\psi}{\partial x_i \partial x_j} + \sum_{j=1}^{n-1} \beta^j(x')\frac{\partial\psi}{\partial x_j} + \gamma(x')\psi$$

Symbol of a Pseudo-Differential Operator (1)

$$-\sum_{i,j=}^{n-1} \alpha^{ij}(x')\xi_i\xi_j - \sqrt{-1}\sum_{j=1}^{n-1}\beta^j(x')\xi_j + \gamma(x')$$

$$+\mu(x')|\xi'|$$

(H) $\mu(x') > 0$ on Γ

$$\sum_{i,j=}^{n-1} \alpha^{ij}(x')\xi_i\xi_j - \gamma(x') \geq 0 \text{ on } T^*(\Gamma)$$

(1) Garding's Inequality

$$\text{(H)} \quad \mu(x') > 0 \text{ on } \Gamma$$

$\exists c_0 > 0,\ \exists c_1 > 0$ such that

$$-\mathrm{Re}\big(\mu(x')\Pi\varphi, \varphi\big) \geq c_0 \|\varphi\|^2_{H^{1/2}(\Gamma)} - c_1 \|\varphi\|^2_{L^2(\Gamma)}$$

Lars Garding

Lars Garding (1919-2014)
Swedish Mathematician

(2) Real Part of the Symbol

The symbol of $-\mathrm{Re}\, Q$

$$= \sum_{i,j=}^{n-1} \alpha^{ij}(x')\xi_i\xi_j - \frac{1}{2}\mathrm{div}\beta(x') - \gamma(x')$$

on $T^*(\Gamma)$

(3) The Fefferman-Phong Inequality

$$\sum_{i,j=}^{N-1} \alpha^{ij}(x')\xi_i\xi_j - \gamma(x') \geq 0 \text{ on } T^*(\Gamma)$$

⇓

$\exists c_2 > 0$ such that

$$-\operatorname{Re}\left(Q\varphi, \varphi\right) \geq -c_2 \|\varphi\|_{L^2(\Gamma)}^2$$

References

- **Fefferman, C. and Phong, D.H.**

On positivity of pseudo-differential operators, **Proc. Nat. Acad. Sci. 75 (1978), 4673-4674.**

(4) The Energy Estimate

$$-\mathrm{Re}\left(T\varphi,\varphi\right)=-\mathrm{Re}\left(\mu(x')\Pi\varphi,\varphi\right)-\mathrm{Re}\left(Q\varphi,\varphi\right)$$

$$\geq c_0\left\|\varphi\right\|^2_{H^{1/2}(\Gamma)}-\left(c_1+c_2\right)\left\|\varphi\right\|^2_{L^2(\Gamma)}$$

$$T=\mu(x')\Pi+Q$$

Hypoellipticity (Hormander)

$$-\mathrm{Re}\left(T\varphi,\varphi\right)=-\mathrm{Re}\left(\mu(x')\Pi\varphi,\varphi\right)-\mathrm{Re}\left(Q\varphi,\varphi\right)$$

$$\geq c_0\left\|\varphi\right\|^2_{H^{1/2}(\Gamma)}-\left(c_1+c_3\right)\left\|\varphi\right\|^2_{L^2(\Gamma)}$$

$$\psi\in D'(\Gamma),\ \ T\psi\in H^{s+1/2}(\Gamma)$$

$$\Rightarrow$$

$$\psi\in H^{s+3/2}(\Gamma)$$

(loss of (3/2 = 1 + 1/2) one derivative)

References

- **Hormander: A Class of Hypoelliptic Pseudodifferential operators with Double Characteristics, Math. Ann. 217 (1975), 165-188.**

Regularity Theorem

Sobolev Regularity Theorem

$$\begin{cases} u \in L^2(\Omega) \\ f = \Delta u \in H^s(\Omega) \end{cases}$$

$$\Rightarrow$$

$$u = G_D f + P\psi \in H^{s+2}(\Omega)$$

The **elliptic gain** of 2-derivatives as in the **Neumann case**

Construction

of

Green Operator

Agmon's Method

Shmuel Agmon

Shmuel Agmon (1922-)
Israel Mathematician

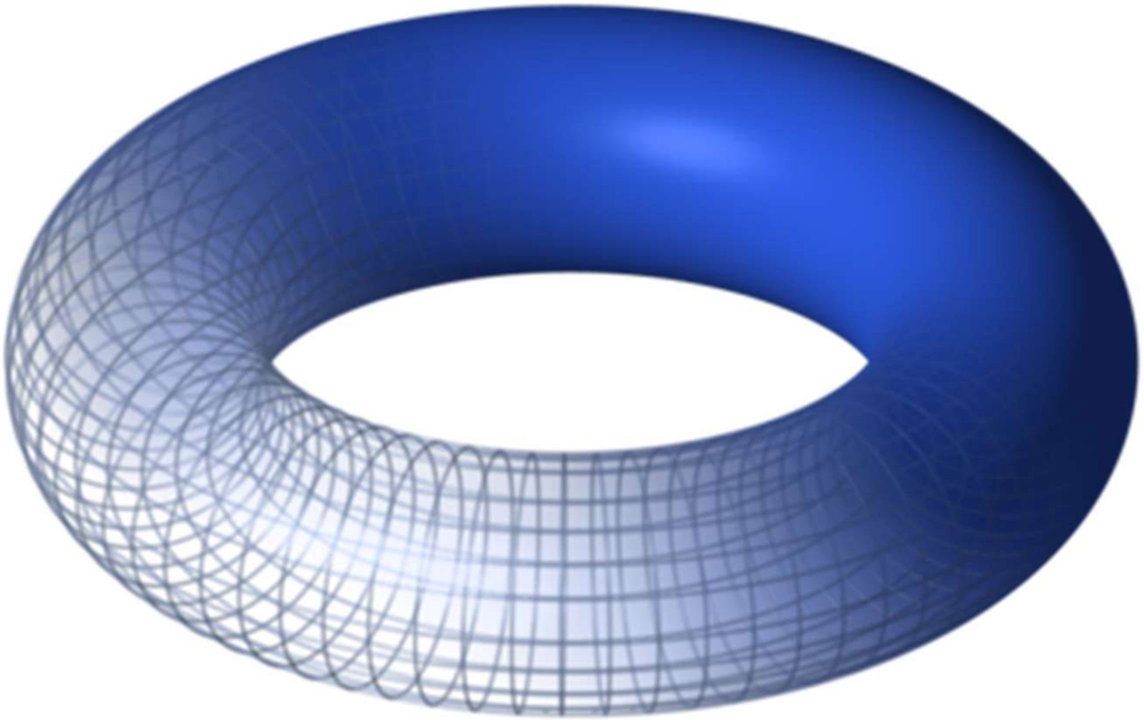
The Idea of Approach

We make use of Agmon's technique of treating a spectral parameter as a second-order elliptic differential operator of an extra variable on the unit circle and relating the old problem to a new one with the additional variable.

References

- **S. Agmon:** **Lectures on elliptic boundary value problems. Van Nostrand, Princeton, New Jersey, 1965.**

Product Domain



$$\Omega \times S^1$$

Differential Operator with a Complex Parameter

$$\Delta u + \lambda I = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \lambda I \quad \text{in } \Omega$$

$$\lambda = r^2 e^{i\theta}$$

$$-\pi < \theta < \pi$$

Augmented Strongly Uniform Elliptic Differential Operator

$$\widetilde{A}(\theta) = \Delta u + e^{i\theta} \frac{\partial^2 u}{\partial y^2} \quad \text{in } \Omega \times S^1$$

$$\widetilde{A}(\theta): \text{ Elliptic}$$

$$\Leftrightarrow$$

$$-\pi < \theta < \pi$$

Augmented Boundary Condition

$$\widetilde{\Lambda}(\theta)\tilde{u} := \mu(x')\frac{\partial \tilde{u}}{\partial \mathbf{n}} + \left(Q\tilde{u}\Big|_{\Gamma\times S} \right) = 0 \quad \text{on } \Gamma\times S$$

Fundamental Hypothesis

(H) $\mu(x') > 0$ on Γ

Augmented Closed Realization

We define a **densely definded, closed** operator

$\widetilde{\mathfrak{A}}_2(\theta) : L^2(\Omega \times S) \rightarrow L^2(\Omega \times S)$

as follows:

(a) $D(\widetilde{\mathfrak{A}}_2(\theta)) = \left\{ \tilde{u} \in H^2(\Omega \times S) : \boxed{\widetilde{\Lambda}(\theta)\tilde{u} = 0 \ \ on \, \Gamma \times S} \right\}$

(b) $\widetilde{\mathfrak{A}}_2(\theta)\tilde{u} = \widetilde{A}(\theta)\tilde{u}, \ \ \forall \tilde{u} \in D(\widetilde{\mathfrak{A}}_2(\theta))$

Then $\widetilde{\mathfrak{A}}_2(\theta)$ is a **Fredholm operator**.

(1) *A priori* estimate

$$\left\|\tilde{u}\right\|_{H^2(\Omega\times S)} \le \tilde{C}(\theta)\left(\left\|\tilde{A}(\theta)\tilde{u}\right\|_{L^2(\Omega\times S)} + \left\|\tilde{u}\right\|_{L^2(\Omega\times S)}\right)$$

$$\forall \tilde{u} \in D(\tilde{\mathfrak{A}}_2(\theta))$$

$$-\pi < \theta < \pi$$

(2) Peetre's Criterion

$$\left\|\tilde{u}\right\|_{H^2(\Omega\times S)} \le \tilde{C}(\theta)\left(\left\|\tilde{A}(\theta)\tilde{u}\right\|_{L^2(\Omega\times S)} + \left\|\tilde{u}\right\|_{L^2(\Omega\times S)}\right)$$

$$\forall \tilde{u} \in D(\widetilde{\mathfrak{A}}_2(\theta))$$

⇓

$$\textbf{ind}\, \widetilde{\mathfrak{A}}_2(\theta) < \infty$$

Index Formula (1981)

$$\mathbf{ind}\,\widetilde{\mathfrak{A}}_2(\theta) < \infty$$

⇓

$$\mathbf{ind}\left(\mathfrak{A}_2 - \lambda I\right) = 0, \quad \forall \lambda \in \mathbf{C}$$

Closed Realization

We define a **densely defined, closed** operator

$$\mathfrak{A}_2 : L^2(\Omega) \to L^2(\Omega)$$

as follows:

(a) $D(\mathfrak{A}_2) = \left\{ u \in H^2(\Omega) : \boxed{\Lambda u = 0 \text{ on } \Gamma} \right\}$

(b) $\mathfrak{A}_2 u = \Delta u, \ \forall u \in D(\mathfrak{A}_2)$

Then $\mathfrak{A}_2 - \lambda I$ is a **Fredholm operator** with **index zero** for $\forall \lambda \in \mathbf{C}$.

Resolvent Estimates

Resolvent Estimates (1)

$$\left\| \tilde{u} \right\|_{H^2(\Omega\times S)} \le \tilde{C}(\theta)\left(\left\| \tilde{A}(\theta)\tilde{u} \right\|_{L^2(\Omega\times S)} + \left\| \tilde{u} \right\|_{L^2(\Omega\times S)} \right)$$

$$\forall \tilde{u} \in D(\tilde{\mathfrak{A}}_2(\theta))$$

$$\Downarrow$$

$$|\lambda| \left\| u \right\|_{L^2(\Omega)} \le \tilde{C}'(\theta) \left\| (\Delta - \lambda) u \right\|_{L^2(\Omega)}$$

Localization function

$$(1)\quad \zeta(y) \in C^{\infty}(S)$$

$$(2)\quad \operatorname{supp}\zeta \subset \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$$

$$(3)\quad \zeta(y) = 1, \quad \forall y \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

Product function

$$\tilde{v}_\eta(x,y) = u(x) \otimes \zeta(y) e^{i\eta y}$$

$$\boxed{u \in D(\mathfrak{A}_2)} \quad \boxed{\eta \geq 0}$$

$$\Rightarrow$$

$$\tilde{v}_\eta(x,y) \in H^2(\Omega \times S)$$

$$\Lambda\left(\tilde{v}_\eta(x,y)\right) = \left(\Lambda u(x)\right) \otimes \zeta(y) e^{i\eta y} = 0$$

$$\boxed{\tilde{v}_\eta(x,y) \in D(\widetilde{\mathfrak{A}}_2(\theta))}$$

***A priori* estimates**

$$\left\|\tilde{v}_{\eta}\right\|_{H^{2}(\Omega\times S)} \leq \widetilde{C}(\theta)\left(\left\|\widetilde{\Lambda}(\theta)\tilde{v}_{\eta}\right\|_{L^{2}(\Omega\times S)} + \left\|\tilde{v}_{\eta}\right\|_{L^{2}(\Omega\times S)}\right)$$

$$\tilde{v}_{\eta}(x,y) = u(x)\otimes\zeta(y)e^{i\eta y} \in D(\widetilde{\mathfrak{A}}_{2}(\theta))$$

Norm estimates (1)

$$\left\| \tilde{v}_{\eta} \right\|_{L^2(\Omega \times S)} = \left\| u(x) \otimes \zeta(y) e^{i\eta y} \right\|_{L^2(\Omega \times S)}$$

$$= \left\| u \right\|_{L^2(\Omega)} \bullet \left\| \zeta \right\|_{L^2(S)}$$

Norm estimates (2)

$$\left\|\widetilde{\Lambda}(\theta)\tilde{v}_\eta\right\|_{L^2(\Omega\times S)} = \left\|\left(A + e^{i\theta}\frac{\partial^2}{\partial y^2}\right)\left(u\otimes\zeta(y)e^{i\eta y}\right)\right\|_{L^2(\Omega\times S)}$$

$$\leq \left\|\left(A - \eta^2 e^{i\theta}\right)u\otimes\zeta(y)e^{i\eta y}\right\|_{L^2(\Omega\times S)}$$

$$+2\eta\left\|u\otimes\zeta'(y)e^{i\eta y}\right\|_{L^2(\Omega\times S)} + \left\|u\otimes\zeta''(y)e^{i\eta y}\right\|_{L^2(\Omega\times S)}$$

$$\leq \|\zeta\|_{L^2(S)}\bullet\left\|\left(A - \eta^2 e^{i\theta}\right)u\right\|_{L^2(\Omega)}$$

$$+\left(2\eta\|\zeta'\|_{L^2(S)} + \|\zeta''\|_{L^2(S)}\right)\|u\|_{L^2(\Omega)}$$

$$\lambda = \eta^2 e^{i\theta}$$

Norm estimates (3)

$$\left\|\tilde{v}_\eta\right\|^2_{H^2(\Omega\times S)} = \left\|u(x)\otimes\zeta(y)e^{i\eta y}\right\|^2_{H^2(\Omega\times S)}$$

$$= \sum_{|\alpha|\le 2}\iint_{\Omega\times S}\left|D^\alpha_{x,y}\left(u(x)\otimes\zeta(y)e^{i\eta y}\right)\right|^2 dxdy$$

$$\ge \sum_{|\alpha|\le 2}\int_\Omega dx\int_{\pi/2}^{3\pi/2}\left|D^\alpha_{x,y}\left(u(x)\otimes e^{i\eta y}\right)\right|^2 dy$$

$$= \sum_{k+|\beta|\le 2}\int_\Omega dx\int_{\pi/2}^{3\pi/2}\left|\eta^k D^\beta u(x)\right|^2 dy$$

Norm estimates (4)

$$\left\|\tilde{v}_\eta\right\|^2_{H^2(\Omega\times S)} = \left\|u(x)\otimes\zeta(y)e^{i\eta y}\right\|^2_{H^2(\Omega\times S)}$$

$$\geq \sum_{k+|\beta|\leq 2}\int_\Omega dx\int_{\pi/2}^{3\pi/2}\left|\eta^k D^\beta u(x)\right|^2 dy$$

$$\geq \pi\left(\sum_{|\beta|=2}\int_\Omega\left|D^\beta u(x)\right|^2 dx+\eta^2\sum_{|\beta|=1}\int_\Omega\left|D^\beta u(x)\right|^2 dx\right.$$

$$\left.+\eta^4\int_\Omega|u(x)|^2 dx\right)$$

Norm estimates (5)

$$|u|_2 + \eta |u|_1 + \eta^2 \|u\|_{L^2(\Omega)}$$

$$\leq \widetilde{C}(\theta)\left(\left\|\left(A - \eta^2 e^{i\theta}\right)u\right\|_{L^2(\Omega)} + \eta \|u\|_{L^2(\Omega)}\right)$$

$$\Rightarrow$$

$$\boxed{|u|_2 + \eta |u|_1 + \eta^2 \|u\|_{L^2(\Omega)} \leq \widetilde{C}'(\theta)\left\|\left(A - \eta^2 e^{i\theta}\right)u\right\|_{L^2(\Omega)}}$$

$$\lambda = \eta^2 e^{i\theta}$$

Resolvent estimates

$$\left\| \tilde{v}_\eta \right\|_{H^2(\Omega\times S)} \leq \widetilde{C}(\theta)\left(\left\| \widetilde{\Lambda}(\theta)\tilde{v}_\eta \right\|_{L^2(\Omega\times S)} + \left\| \tilde{v}_\eta \right\|_{L^2(\Omega\times S)} \right)$$

$$\left| u \right|_2 + \sqrt{\left|\lambda\right|}\left| u \right|_1 + \left|\lambda\right| \left\| u \right\|_{L^2(\Omega)} \leq \widetilde{C}'(\theta) \left\| (A-\lambda)u \right\|_{L^2(\Omega)}$$

$$\lambda = \eta^2 e^{i\theta}$$

Fredholm Alternative

The **Fredholm alternative**

holds true for the closed operator

$$\mathfrak{A}_2 - \lambda I, \ \forall \lambda \in \mathbf{C}$$

Resolvent Estimates (2)

$$|\lambda| \|u\|_{L^2(\Omega)} \leq \widetilde{C}'(\theta) \|(\Delta - \lambda)u\|_{L^2(\Omega)}$$

$$\mathbf{ind}\left(\mathfrak{A}_2 - \lambda I\right) = 0, \quad \forall \lambda \in \mathbf{C}$$

⇓

$$\left\|\left(\mathfrak{A}_2 - \lambda I\right)^{-1} f\right\|_{L^2(\Omega)} \leq \frac{\widetilde{C}'(\theta)}{|\lambda|} \|f\|_{L^2(\Omega)}$$

$$\forall f \in L^2(\Omega)$$

Asymptotic eigenvalue distribution for Vishik-Wentzell Boundary Value Problems

Main Idea
via
the Resolvent

Essential Points

(1) The **degenerate** case:

We cannot use **Green's formula** to characterize the **adjoint operator** $\mathfrak{A}_2^*$.

(2) We shift our attention to the **resolvent** $\mathfrak{A}_2^{*-1}$, instead of $\mathfrak{A}_2^*$.

Boutet de Monvel Calculus

Louis Boutet de Monvel

◆Louis Boutet de Monvel (1941-2014)
French Mathematician

References

- **Boutet de Monvel: Boundary problems for pseudo-differential operators. Acta Matematica, 126 (1971), 11-51.**

Boutet de Monvel Calculus (General form)

$$\mathfrak{A} = \begin{pmatrix} A + G & K \\ T & Q \end{pmatrix}$$

Representation Formula
of the Resolvent

Reduction to the Boundary (1)

Consider the Vishik - Wentzell Boundary Value problem

$$\begin{cases} Au = f \quad \text{in } \Omega, \\ \Lambda u = \mu(x')\dfrac{\partial u}{\partial \boldsymbol{n}} + Q\left(u\big|_{\Gamma}\right) = 0 \quad \text{on } \Gamma \end{cases}$$

Reduction to the Boundary (2)

Solve the Dirichlet problem

$$\begin{cases} Av = f \in H^s(\Omega), \\ v = 0 \text{ on } \Gamma. \end{cases}$$

We let

$$v := \exists! G_D f \in H^{s+2}(\Omega)$$

Reduction to the Boundary (3)

Let

$$w := u - v = u - G_D f$$

Reduction to the Boundary (4)

$$\Lambda v = \Lambda\left(G_D f\right)$$

$$= \mu(x')\frac{\partial v}{\partial \boldsymbol{n}} + Q\left(v\big|_{\Gamma}\right) = \mu(x')\frac{\partial v}{\partial \boldsymbol{n}}$$

$$= \mu(x')\gamma_1\left(G_D f\right) \in H^{s+1/2}(\Gamma)$$

Reduction to the Boundary (5)

Then

$$\begin{cases} Au = f & \text{in } \Omega, \\ \Lambda u = 0 & \text{on } \Gamma \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} Aw = Au - Av = 0 & \text{in } \Omega, \\ \Lambda w = -\Lambda v = -\mu(x')\gamma_1(G_D f) & \text{on } \Gamma \end{cases}$$

Reduction to the Boundary (6)

$$\Lambda w = \Lambda u - \Lambda v$$

$$= -\mu(x')\gamma_1\left(G_D f\right) \in H^{s+1/2}\left(\Gamma\right)$$

(gain of one derivative)

Reduction to the Boundary (7)

$$Aw = 0 \ \text{ in } \Omega$$

$$\Leftrightarrow$$

$$w = P\psi$$ **(Poisson operator)**

Reduction to the Boundary (8)

$$\begin{cases} Au = f & \text{in } \Omega, \\ \Lambda u = 0 & \text{on } \Gamma \end{cases}$$

$$\Leftrightarrow$$

$$\left(\Lambda P\right)\psi = Bw$$

$$= -\mu(x')\gamma_1\left(G_D f\right) \text{ on } \Gamma$$

Fredholm Boundary Operator

$$
\begin{aligned}
(\Lambda P)\psi &= \mu(x')\frac{\partial}{\partial\nu}(P\psi)\Big|_{\Gamma} + Q\psi \\
&= \mu(x')\Pi\psi \\
&\quad + \sum_{i,j=1}^{n-1}\alpha^{ij}(x')\frac{\partial^2\psi}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1}\beta^{i}(x')\frac{\partial\psi}{\partial x_i} + \gamma(x')\psi
\end{aligned}
$$

Differential Operators

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$$

$$A := (1 - \Delta)u = u - \left(\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \right)$$

Bessel Potential of order 2 (1)

$$G_2 * f(x) = \int_{\mathbf{R}^n} G_2(x-y) f(y)\, dy$$

Bessel Potential of order 2 (2)

$$G_2(x) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty e^{-t-\frac{|x|^2}{4t}} t^{\frac{2-n}{2}} \frac{dt}{t}$$

$$\widehat{G}_2(\xi) = \int_{\mathbf{R}^n} e^{-ix\xi} G_2(x)\,dx = \frac{1}{|\xi|^2 + 1}$$

Zero Extension Operator

$$e^{+} f(x) = f_0(x)$$

$$:= \begin{cases} f(x) & \forall x \in \Omega, \\ 0 & \forall x \notin \Omega \end{cases}$$

$$\left(G_2 * f_0\right)(x)$$

$$= \int_{\Omega} G_2(x-y)\, f(y)\, dy$$

Transmission property of Bessel Potential (Boutet de Monvel)

$$\left(r^+ G_2\right) f := \left(G_2 * f_0\right)\Big|_\Omega$$

$$r^+ G_2 : H^s(\Omega) \to H^{s+2}(\Omega)$$

Right Inverse Operator

$$(1-\Delta)\left(r^{+}G_{2}f\right)=f \quad \text{in } \Omega$$

$$(1-\Delta)u = u - \left(\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}\right)$$

Reduction to the boundary (1)

$$(1-\Delta)u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma$$

$$\Rightarrow$$

$$(1-\Delta)\left(r^{+}G_{2}f - u\right) = f - f = 0 \quad \text{in } \Omega$$

Poisson Kernel (1)

$$(1 - \Delta)(P_2\varphi) = 0 \quad \text{in } \Omega,$$
$$P_2\varphi\,\big|_{\Gamma} = \varphi \quad \text{on } \Gamma$$

Poisson Kernel (2)

$$P_2 : H^{s-1/2}(\Gamma) \to H^s(\Omega)$$

P_2

Potential Symbol: $\dfrac{1}{\langle \xi' \rangle + i\nu}$,

$$\xi = (\xi', \nu), \quad \langle \xi' \rangle = \sqrt{1 + |\xi'|^2}$$

Poisson Kernel in the Half-Space (1)

$$P_2\varphi(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbf{R}^{n-1}} e^{ix'\xi'} e^{-x_n\langle\xi'\rangle} \hat{\varphi}(\xi')\, d\xi'$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix_n\nu} \frac{1}{\langle\xi'\rangle + i\nu}\, d\nu = e^{-x_n\langle\xi'\rangle}$$

Poisson Kernel in the Half-Space (2)

$$P_2\varphi(x',x_n)$$

$$=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ix'\xi'+ix_n\nu}\frac{1}{\langle\xi'\rangle+i\nu}\hat{\varphi}(\xi')\,d\xi'\,d\nu$$

$$=\frac{1}{(2\pi)^{n-1}}\int_{\mathbf{R}^{n-1}} e^{ix'\xi'}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix_n\nu}\frac{1}{\langle\xi'\rangle+i\nu}\,d\nu\right)\hat{\varphi}(\xi')\,d\xi'$$

$$=\frac{1}{(2\pi)^{n-1}}\int_{\mathbf{R}^{n-1}} e^{ix'\xi'}e^{-x_n\langle\xi'\rangle}\hat{\varphi}(\xi')\,d\xi'$$

Reduction to the boundary (2)

$$r^{+} G_{2} f - u = P_{2} \psi$$

$$\psi = \left(r^{+} G_{2} f - u \right) \Big|_{\Gamma}$$

Reduction to the boundary (3)

$$u = r^+ G_2 f + P_2 \psi$$
$$u = 0$$

$$\Leftrightarrow$$

$$\psi = \left(u - r^+ G_2 f \right) \text{ on } \Gamma$$

Dirichlet-to-Neumann Operator

$$\Pi_2 \psi = \frac{\partial}{\partial \mathbf{n}} \left(P_2 \psi \right) \Big|_{\Gamma}, \quad \forall \psi \in C^{\infty}(\Gamma)$$

$$\Pi_2 \in L^1_{\mathrm{cl}}(\Gamma)$$

Unique Solvability of Dirichlet Problem

$$\begin{cases} (1-\Delta)u = f & \text{in } \Omega \\ u = \varphi & \text{on } \Gamma \end{cases}$$

$$\Longleftrightarrow$$

$$u = r^+ G_2 f + P_2\left(\varphi - \left(r^+ G_2 f\right)\Big|_\Gamma\right)$$

Green Operator for the Dirichlet Problem

$$(1-\Delta)v = f \quad \text{in } \Omega,$$
$$v = 0 \quad \text{on } \Gamma$$

$\Leftrightarrow$

$$v = G_D f$$
$$= r^+ G_2 f - P_2\left(\gamma_0\left(r^+ G_2 f\right)\right)$$

Boutet de Monvel Calculus (1)

$$\mathfrak{C} = \begin{pmatrix} r^+ G_2 & -P_2 \\ \gamma_0 \left(r^+ G_2 \right) & I \end{pmatrix}$$

Symbolic Calculus (1)

$$\begin{pmatrix} -\dfrac{1}{2\langle \xi' \rangle}\left(\dfrac{1}{\langle \xi' \rangle + i\nu} + \dfrac{1}{\langle \xi' \rangle - i\nu}\right) & -\dfrac{1}{\langle \xi' \rangle + i\nu} \\ -\dfrac{1}{2\langle \xi' \rangle}\dfrac{1}{\langle \xi' \rangle - i\tau} & 1 \end{pmatrix}$$

Boutet de Monvel Calculus (2)

$$\mathfrak{C}: \begin{array}{c} H^{s}(\Omega) \\ \oplus \\ H^{s+1/2}(\Gamma) \end{array} \to \begin{array}{c} H^{s+2}(\Omega) \\ \oplus \\ B^{s+1/2}(\Gamma) \end{array}$$

Mapping Property (1)

$$
\begin{array}{ccc}
H^{s}(\Omega) & \xrightarrow{\gamma_0\left(r^{+}G_2\right)} & H^{s+3/2}(\partial\Omega) \\
r^{+}G_2 \downarrow & & \downarrow I \\
H^{s+2}(\Omega) & \xleftarrow[-P_2]{} & H^{s+3/2}(\partial\Omega)
\end{array}
$$

$$G_D = r^{+}G_2 - P_2\left(\gamma_0\left(r^{+}G_2\right)\right)$$

Boutet de Monvel Calculus (3)

$$\mathfrak{C}^* = \begin{pmatrix} \left(r^+ G_2\right)^* & \left(\gamma_0 \left(r^+ G_2\right)\right)^* \\ -P_2^* & I \end{pmatrix}$$

$$G_D{}^* = \left(r^+ G_2\right)^* - \left(\gamma_0 \left(r^+ G_2\right)\right)^* P_2{}^*$$

Symbolic Calculus (2)

$$\begin{pmatrix} -\frac{1}{2\langle \xi' \rangle}\left(\frac{1}{\langle \xi' \rangle + i\nu} + \frac{1}{\langle \xi' \rangle - i\nu}\right) & -\frac{1}{2\langle \xi' \rangle}\frac{1}{\langle \xi' \rangle + i\nu} \\ -\frac{1}{\langle \xi' \rangle - i\tau} & 1 \end{pmatrix}$$

Boutet de Monvel Calculus (4)

$$\mathfrak{C}^* : \begin{array}{ccc} H^s(\Omega) & & H^{s+2}(\Omega) \\ \oplus & \rightarrow & \oplus \\ H^{s+1/2}(\Gamma) & & H^{s+1/2}(\Gamma) \end{array}$$

Mapping Property (2)

$$
\begin{array}{ccc}
H^{s}(\Omega) & \xrightarrow{-P_2^*} & H^{s+1/2}(\Gamma) \\
(r^+G_2)^* \downarrow & & \downarrow I \\
H^{s+2}(\Omega) & \xleftarrow[\left(\gamma_0 (r^+G_2)\right)^*]{} & H^{s+1/2}(\Gamma)
\end{array}
$$

$$G_D^* = \left(r^+ G_2\right)^* - \left(\gamma_0 \left(r^+ G_2\right)\right)^* P_2^*$$

Representation of the Resolvent

$$u = G_D f - P_2\left(T_2^{-1}\left(\Lambda G_D f\right)\right)$$

$$T_2^{-1} \in L_{1/2,1/2}^{-1}\left(\Gamma\right)$$

$$T_2 = \Lambda P_2 = Q + \mu(x')\Pi_2 \in L_{1,0}^2\left(\Gamma\right)$$

$$-\sum_{i,j=1}^{n-1} \alpha^{ij}(x')\xi_i\xi_j - \mu(x')\left|\xi'\right|$$

Boutet de Monvel Calculus (5)

$$\mathfrak{E} = \begin{pmatrix} G_D & -P_2 \\ \Lambda G_D & (\Lambda P_2)^{-1} \end{pmatrix}$$

$$\Lambda G_D = \mu(x')\gamma_1 G_D$$

$$-\mu(x')\frac{1}{\langle \xi' \rangle - i\tau}$$

Symbolic Calculus (3)

$$\begin{pmatrix} \sigma\left(G_D\right) & -\dfrac{1}{\langle \xi' \rangle + i\nu} \\ -\dfrac{\mu(x')}{\langle \xi' \rangle - i\tau} & \sigma\left(T_2^{-1}\right) \end{pmatrix}$$

Mapping Property (1)

$$\begin{array}{ccc} H^{s}(\Omega) & \xrightarrow{\mu(x')\gamma_1 G_D} & H^{s+1/2}(\Gamma) \\ G_D \downarrow & & \downarrow T_2^{-1} \\ H^{s+2}(\Omega) & \xleftarrow[-P_2]{} & H^{s+3/2}(\Gamma) \end{array}$$

$$G = G_D - P_2\left(T_2^{-1}\left(\mu(x')\gamma_1 G_D\right)\right)$$

Regularity of the Resolvent

$$\left(I-\mathfrak{A}_2\right)^{-1}: H^s(\Omega) \rightarrow H^{s+2}(\Omega)$$

$$\left(I-\mathfrak{A}_2\right)^{-k}: L^2(\Omega) \rightarrow H^{2k}(\Omega), \quad \forall k \in \mathbf{N}$$

Mapping Property (2)

$$
\begin{array}{ccc}
H^{s}(\Omega) & \xrightarrow{-P_2^*} & H^{s+1/2}(\Gamma) \\
G_D^* \downarrow & & \downarrow \left(T_2^{-1}\right)^* \\
H^{s+2}(\Omega) & \xleftarrow[\mu(x')\left(\gamma_1 G_D\right)^*]{} & H^{s+3/2}(\Gamma)
\end{array}
$$

$$G^* = G_D^* - \mu(x')\left(\gamma_1 G_D\right)^* \left(T_2^{-1}\right)^* P_2^*$$

Regularity of the Adjoint of the Resolvent

$$\left(I-\mathfrak{A}_2^*\right)^{-1}: H^s(\Omega) \rightarrow H^{s+2}(\Omega)$$

⇓

$$\left(I-\mathfrak{A}_2^*\right)^{-k}: L^2(\Omega) \rightarrow H^{2k}(\Omega), \quad \forall k \in \mathbf{N}$$

END