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Cut-and-Project Schemes for Pisot Family Substitution Tilings

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Abstract: We consider Pisot family substitution tilings in $\mathbb{R}^d$ whose dynamical spectrum is pure point. There are two cut-and-project schemes (CPSs) which arise naturally: one from the Pisot family property and the other from the pure point spectrum. The first CPS has an internal space $\mathbb{R}^m$ for some integer $m \in \mathbb{N}$ defined from the Pisot family property, and the second CPS has an internal space H that is an abstract space defined from the condition of the pure point spectrum. However, it is not known how these two CPSs are related. Here we provide a sufficient condition to make a connection between the two CPSs. For Pisot unimodular substitution tiling in $\mathbb{R}$, the two CPSs turn out to be same due to the remark by Barge-Kwapisz.

Keywords: Pisot substitution tilings; pure point spectrum; regular model set; algebraic coincidence

1. Introduction

The study of long-range aperiodic order has played an important role in understanding the structures of physical models like "quasicrystals". As an important class of models for the long-range aperiodic order, substitution tilings have been the subject of a great deal of study. Among these substitution tilings, Pisot or Pisot family substitutions get more attention due to their well-ordered properties. A Pisot number is an algebraic integer $\lambda > 1$ for which all of the other algebraic conjugates of λ lie strictly inside the unit circle. If the largest eigenvalue of the corresponding substitution matrix is a Pisot number, we call the substitution a Pisot substitution. There is a well-known conjecture called the "Pisot substitution conjecture". The associated substitution system always has a pure point spectrum [1]. For special cases of the Pisot substitutions, the conjecture has been answered affirmatively [2,3]. Extending the idea of a Pisot number, one can also look at a Pisot family, a set of algebraic integers whose Galois conjugates with modulus larger than or equal to 1 are all in the same set [4]. There has been a great deal of of study on Pisot substitution sequences or Pisot family substitution tilings on $\mathbb{R}^d$ which characterizes the property of pure point spectrum (see [1] and therein). There are two natural cut-and-project schemes (CPSs) arising in this study. One CPS is constructed with a Euclidean internal space using the Pisot property [5]. We extend the idea of constructing the CPS from the Pisot property to the Pisot family property. The other CPS is made by constructing an abstract internal space from the property of pure point spectrum [6]. These two CPSs were developed independently from different aims of study. It is not yet known if these two CPSs have any relation to each other. Here we would like to provide how these two CPSs are related, showing that two internal spaces are basically isomorphic to each other in Theorem 2.

In this article, we consider mainly primitive substitution tilings in $\mathbb{R}^d$ with pure point spectrum. It is proven in [7] that primitive substitution tilings with pure point spectrum always show finite local complexity (FLC). So, it is not necessary to make an assumption of FLC in the consideration of pure point spectrum.

In Section 2, we visit the basic definitions of the terms that we use. In Section 3, we construct a natural CPS which arises from the property of Pisot family substitution. In Section 4, we recall the other CPS constructed from the property of pure point spectrum. We show in Theorem 2 that the two CPSs are closely related, demonstrating that there is an isomorphism between two internal spaces of the CPSs under certain model set conditions. In Section 6, we raise a few questions for later study.

2. Preliminary

2.1. Tilings

We consider a set of types (or colours) $\{1,\ldots,\kappa\}$. A *tile* in $\mathbb{R}^d$ is a pair $T = (A, i)$ where $A = \mathrm{supp}(T)$ (the support of T) is a compact set in $\mathbb{R}^d$ with $A = \overline{A^\circ}$ and $i = l(T) \in \{1,\ldots,\kappa\}$ is the type of T. Let $h + T = (h + A, i)$ for $h \in \mathbb{R}^d$. A set P of tiles is a *patch* if the number of tiles in P is finite and the interiors of the tiles are mutually disjoint. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is a set of tiles for which $\mathbb{R}^d = \bigcup\{\mathrm{supp}(T) : T \in \mathcal{T}\}$ and the interiors of distinct tiles are disjoint. Given a tiling $\mathcal{T}$, $\mathcal{T}$*-patch* is a finite set of tiles of $\mathcal{T}$. We always assume that any two $\mathcal{T}$-tiles with the same type are translationally equivalent. Thus, up to translations, there is a finite number of $\mathcal{T}$-tiles.

We will use the following notation:

$$K^{+s} := \{x \in \mathbb{R}^d : \mathrm{dist}(x, K) \le s\} \text{ and } K^{-s} := \{x \in K : \mathrm{dist}(x, \partial K) \ge s\}. \tag{1}$$

A *van Hove sequence* for $\mathbb{R}^d$ is a sequence $\mathcal{K} = \{K_n\}_{n\ge 1}$ of bounded measurable subsets of $\mathbb{R}^d$ satisfying

$$\lim_{n\to\infty} \mathrm{Vol}((\partial K_n)^{+s})/\mathrm{Vol}(K_n) = 0, \text{ for all } s > 0. \tag{2}$$

2.2. Delone κ-Sets

A Delone set in $\mathbb{R}^d$ is a point set which is relatively dense and uniformly discrete in $\mathbb{R}^d$. We call $\mathbf{\Lambda} = (\Lambda_i)_{i\le\kappa}$ a *Delone κ-set* in $\mathbb{R}^d$ if each Λ_i is Delone and $\mathrm{supp}(\mathbf{\Lambda}) := \bigcup_{i=1}^{\kappa} \Lambda_i \subset \mathbb{R}^d$ is Delone. A Delone κ-set $\mathbf{\Lambda} = (\Lambda_i)_{i\le\kappa}$ is called *representable* if there exist tiles $T_i = (A_i, i), i \le \kappa$, so that $\{x + T_i : x \in \Lambda_i, i \le \kappa\}$ is a tiling of $\mathbb{R}^d$.

2.3. Substitutions

Definition 1. *Let us consider a finite set $\mathcal{A} = \{T_1,\ldots,T_\kappa\}$ of tiles in $\mathbb{R}^d$ with $T_i = (A_i, i)$, $1 \le i \le \kappa$ which we will call* prototiles. *We denote by $\mathcal{P}_{\mathcal{A}}$ the set of patches which are formed by tiles that are translates of T_i's. $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is a tile-substitution (or substitution) with expansion map ϕ if there are finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le \kappa$, for which*

$$\omega(T_j) = \{x + T_i : x \in \mathcal{D}_{ij}, i = 1,\ldots,\kappa\} \tag{3}$$

with

$$\phi A_j = \bigcup_{i=1}^{\kappa} (\mathcal{D}_{ij} + A_i) \quad \text{for } j \le \kappa. \tag{4}$$

Here all sets in the union have disjoint interiors. It is possible for some of the $\mathcal{D}_{ij}$ to be empty. The substitution $\kappa \times \kappa$ matrix S is defined by $\mathsf{S}(i, j) = \#\mathcal{D}_{ij}$. If $\mathsf{S}^m > 0$ for some $m \in \mathbb{N}$, then the corresponding substitution tiling $\mathcal{T}$ is primitive.

A set of algebraic integers $\Theta = \{\theta_1, \cdots, \theta_r\}$ is a *Pisot family* if for any $1 \leq j \leq r$. Every Galois conjugate γ of θ_j, with $|\gamma| \geq 1$, is contained in Θ. For $r = 1$, with θ_1 real and $|\theta_1| > 1$, this reduces to $|\theta_1|$ being a real Pisot number, and for $r = 2$, with θ_1 non-real and $|\theta_1| > 1$, to θ_1 being a complex Pisot number. We say that $\mathcal{T}$ is a *Pisot substitution tiling* if the expansive map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a Pisot expansive factor λ, and a *Pisot family substitution tiling* if the eigenvalues of the expansive map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ form a Pisot family.

2.4. Cut-and-Project Scheme

Definition 2. A cut and project scheme (CPS) is a collection of spaces and mappings as follows:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \bigcup & & \\ & & \widetilde{L} & & \end{array} \tag{5}$$

where $\mathbb{R}^d$ is a real Euclidean space, H is a locally compact Abelian group, π_1 and π_2 are the canonical projections, $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times H$ (i.e., a discrete subgroup such that the quotient group $(\mathbb{R}^d \times H)/\widetilde{L}$ is compact), $\pi_1|_{\widetilde{L}}$ is injective, and $\pi_2(\widetilde{L})$ is dense in H.

For a subset $V \subset H$, we denote $\Lambda(V) := \{\pi_1(x) \in \mathbb{R}^d : x \in \widetilde{L}, \pi_2(x) \in V\}$.

We define a model set in $\mathbb{R}^d$ as a subset Λ in $\mathbb{R}^d$ for which, up to translation, $\Lambda(W^\circ) \subset \Gamma \subset \Lambda(W)$, W is compact in H, $W = \overline{W^\circ} \neq \emptyset$. If the boundary $\partial W = W \backslash W^\circ$ of W is of (Haar) measure 0, we say that the model set Λ is regular. We say that $\mathbf{\Lambda} = (\Lambda_i)_{i \leq \kappa}$ is a model κ-set (resp. regular model κ-set) if each Λ_i is a model set (resp. regular model set) with regard to the same CPS.

We assume, without loss of generality, that H is generated by the windows W_i's, where $\Lambda_i = \Lambda(W_i)$ for all $i \leq \kappa$. When H satisfies the following:

$$\{s \in H : s + W_i = W_i \text{ for all } i \leq \kappa\} = \{0\}\,, \tag{6}$$

we say that the windows W_i's have irredundancy.

2.5. Pure Point Spectrum

Let $X_{\mathcal{T}}$ be the set of all primitive substitution tilings in $\mathbb{R}^d$ such that clusters of each tiling are translates of a $\mathcal{T}$-patch. We give a usual metric δ in tilings in such a way that two tilings are close if there is a large agreement on a big area with small shift (see [8–10]). Then, $X_{\mathcal{T}} = \overline{\{-h + \mathcal{T} : h \in \mathbb{R}^d\}}$, where we take the closure in the topology induced by the metric δ. We consider a natural action of $\mathbb{R}^d$ by translations on the dynamical hull $X_{\mathcal{T}}$ of $\mathcal{T}$ and get a topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$. Assume that $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ is a measure-preserving dynamical system with a unique ergodic measure μ. We look at the associated group of unitary operators $\{T_x\}_{x \in \mathbb{R}^d}$ on $L^2(X_{\mathcal{T}}, \mu)$:

$$T_x g(\mathcal{T}') = g(-x + \mathcal{T}').$$

Every $g \in L^2(X_{\mathcal{T}}, \mu)$ defines a function on $\mathbb{R}^d$ by $x \mapsto \langle T_x g, g \rangle$, which is positive definite on $\mathbb{R}^d$. So, its Fourier transform is a positive measure σ_g on $\mathbb{R}^d$ and we call it the *spectral measure* corresponding to g. We say that the dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ has *pure point spectrum* if σ_g is pure point for each $g \in L^2(X_{\mathcal{T}}, \mu)$. If the dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ has a pure point spectrum, we also say that $\mathcal{T}$ has a pure point spectrum.

3. Cut-and-Project Scheme for Pisot Family Substitution Tilings

We consider a primitive substitution tiling $\mathcal{T}$ on $\mathbb{R}^d$ with expansion map ϕ. There is a standard way to choose distinguished points in the tiles of primitive substitution tiling so that they form a ϕ-invariant Delone set. They are called *control points*. A tiling $\mathcal{T}$ is called a fixed point of the substitution ω if $\omega(\mathcal{T}) = \mathcal{T}$.

Definition 3 ([11,12])**.** *Let $\mathcal{T}$ be a fixed point of a primitive substitution with expansion map ϕ. For every $\mathcal{T}$-tile T, we choose a tile γT on the patch $\omega(T)$. For all tiles of the same type, we choose γT with the same relative position. This defines a map $\gamma : \mathcal{T} \to \mathcal{T}$ called the tile map. Then, we define the control point for a tile $T \in \mathcal{T}$ by*

$$\{c(T)\} = \bigcap_{m=0}^{\infty} \phi^{-m}(\gamma^m T).$$

The control points satisfy the following:

(a) $T' = T + c(T') - c(T)$, for any tiles T, T' of the same type;
(b) $\phi(c(T)) = c(\gamma T)$, for $T \in \mathcal{T}$.

For tiles of any tiling $\mathcal{S} \in X_{\mathcal{T}}$, control points have the same relative position as in $\mathcal{T}$-tiles. The choice of control points is non-unique, but there are only finitely many possibilities, determined by the choice of the tile map. Let

$$\mathcal{C} := \mathcal{C}(\mathcal{T}) = \{c(T) : T \in \mathcal{T}\}$$

be a set of control points of the tiling $\mathcal{T}$ in $\mathbb{R}^d$. Let

$$\Xi := \Xi(\mathcal{T}) = \bigcup_{i=1}^{\kappa} (\mathcal{C}_i - \mathcal{C}_i),$$

where $\mathcal{C}_i$ is the set of control points of tiles of type i.

Let us assume that ϕ is diagonalizable over $\mathbb{C}$ and the eigenvalues of ϕ are algebraically conjugate with multiplicity one. For a complex eigenvalue λ of ϕ, the 2×2 diagonal block $\begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}$ is similar to a real 2×2 matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = S^{-1} \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} S, \tag{7}$$

where $\lambda = a + ib$, $a, b \in \mathbb{R}$, and $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$. So we can assume, by appropriate choice of basis, that ϕ is diagonal with the diagonal entries equal to λ corresponding to real eigenvalues, and diagonal 2×2 blocks of the form $\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$ corresponding to complex eigenvalues $a_j + ib_j$.

We assume, without loss of generality, that ϕ is a diagonal matrix.

We recall the following theorem. The theorem is not in the form as shown here, but one can readily note that from the proof of [4] (Theorem 4.1).

Theorem 1 ([4] Theorem 4.1)**.** *Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with expansion map ϕ. We assume that $\mathcal{T}$ has FLC, ϕ is diagonalizable, and all the eigenvalues of ϕ are algebraically conjugate with multiplicity one. Then, there exists an isomorphism $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$\sigma\phi = \phi\sigma \quad \text{and} \quad k \cdot \mathbb{Z}[\phi]\alpha \subset \sigma(\mathcal{C}(\mathcal{T})) \subset \mathbb{Z}[\phi]\alpha\,,$$

where $\alpha = (1, 1, \ldots, 1) \in \mathbb{R}^d$ and $k \in \mathbb{Z}$.

Let us assume now that $\mathcal{T}$ has FLC, ϕ is diagonalizable, the eigenvalues of ϕ are all algebraically conjugate with multiplicity one, and there exists at least one other algebraic conjugate different from eigenvalues of ϕ. Suppose that ϕ has e number of real eigenvalues, and f number of 2×2 blocks of the form of complex eigenvalues where $d = e + 2f$. Let all the algebraic conjugates of eigenvalues of ϕ be real numbers $\lambda_1, \ldots, \lambda_s$ and complex numbers $\lambda_{s+1}, \overline{\lambda}_{s+1}, \ldots, \lambda_{s+t}, \overline{\lambda}_{s+t}$. Let $m := s + 2t$ and write $\lambda_{s+t+i} = \overline{\lambda}_{s+i}$ for $i = 1, \ldots t$ for convenience. Let us consider a space $\mathbb{K}$, where

$$\mathbb{K} := \mathbb{R}^{s-e} \times \mathbb{C}^{t-f} \simeq \mathbb{R}^{m-d}.$$

Let us consider the following map:

$$\Psi : \mathbb{Z}[\phi]\xi \to \mathbb{K}, \tag{8}$$

$$P(\phi)\xi \mapsto (P(\lambda_{e+1}), \ldots, P(\lambda_s), P(\lambda_{s+f+1}), \ldots, P(\lambda_{s+t})), \tag{9}$$

where $P(x)$ is a polynomial over $\mathbb{Z}$. Let us build a new cut and project scheme:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{K} & \xrightarrow{\pi_2} & \mathbb{K} \\ & & \cup & & \\ L & \longleftarrow & \widetilde{L} & \longrightarrow & \Psi(L) \\ & & & & \\ x & \longleftarrow\!\shortmid & (x, \Psi(x)) & \longmapsto & \Psi(x)\,, \end{array} \tag{10}$$

where π_1 and π_2 are canonical projections, $L = \langle \mathcal{C}_i \rangle_{i \leq \kappa}$, and $\widetilde{L} = \{(x, \Psi(x)) : x \in L\}$. It is clear to see that $\pi_1|_{\widetilde{L}}$ is injective. We now show that $\pi_2(\widetilde{L})$ is dense in $\mathbb{K}$ and $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times \mathbb{K}$.

Lemma 1. *$\widetilde{L}$ is a lattice in $\mathbb{R}^d \times \mathbb{K}$.*

Proof. Since $\mathbb{Z}[\phi]\alpha$ is a free $\mathbb{Z}$-module of rank m and $m \times m$ matrix $A = (\lambda_i^{j-1})_{i,j \in \{1,\ldots,m\}}$ is non-degenerate by the Vandermonde determinant, the natural embedding combining all conjugates; $f : \mathbb{Z}[\phi]\alpha \to \mathbb{R}^s \times \mathbb{C}^t \simeq \mathbb{R}^d \times \mathbb{K}$ gives a lattice $f(\mathbb{Z}[\phi]\alpha)$ in $\mathbb{R}^d \times \mathbb{K}$. Consequently, $\widetilde{L}$ is isomorphic to a free $\mathbb{Z}$-submodule of $f(\mathbb{Z}[\phi]\alpha)$ due to the theory of elementary divisors. From Theorem 1, $\widetilde{L}$ is isomorphic to a full rank $\mathbb{Z}$-submodule of $f(\mathbb{Z}[\phi]\alpha)$, that is, a sub-lattice of $f(\mathbb{Z}[\phi]\alpha)$. Thus, the claim is shown. The case with complex conjugates can be shown in a similar manner, taking care of embeddings $\mathbb{C}$ to $\mathbb{R}^2$. □

Lemma 2. *$\Psi(L) = \pi_2(\widetilde{L})$ is dense in $\mathbb{K}$.*

Proof. We showed that $\widetilde{L}$ is a sub-lattice of $f(\mathbb{Z}[\phi]\alpha)$ in the proof of Lemma 1. So it suffices to prove that $\Psi(\mathbb{Z}[\phi])$ is dense in $\mathbb{K}$. We prove the totally real case, that is, $\lambda_i \in \mathbb{R}$ for all i. By [13] (Theorem 24), $\Psi(\mathbb{Z}[\phi])$ is dense if

$$\sum_{i=d+1}^{m} x_i \lambda_i^{j-1} \in \mathbb{Z} \quad (j = 1, \ldots, m)$$

implies $x_i = 0$ for $i = d+1, \ldots, m$. The condition is equivalent to

$$\xi A \in \mathbb{Z}^m,$$

with $\xi = (x_i) = (0, \ldots, 0, x_{d+1}, \ldots, x_m) \in \mathbb{R}^m$ in the terminology of Lemma 1. Multiplying the inverse of A, we see that entries of ξ must be Galois conjugates. As ξ has at least one zero entry, we obtain $\xi = 0$, which shows $x_i = 0$ for $i = d+1, \ldots, m$. In fact, this discussion uses the Pontryagin duality that the $\Psi : \mathbb{Z}^m \to \mathbb{R}^{m-d}$ has a dense image if and only if its dual map $\widehat{\Psi} : \mathbb{R}^{m-d} \to \mathbb{T}^m$ is injective (see also [14–16] [Chapter II, Section 1]). The case with complex conjugates is similar. □

4. Two Cut-and-Project Schemes

4.1. ϕ-Topology

Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with expansion map ϕ. Let

$$L := \langle \mathcal{C}_i \rangle_{i \leq \kappa}$$

be the group generated by $\mathcal{C}_i$, $i \leq \kappa$, where $\mathcal{C} = (\mathcal{C}_i)_{i \leq \kappa}$ is a control point set of $\mathcal{T}$ and

$$\mathcal{K} := \{t \in \mathbb{R}^d : \mathcal{T} + t = \mathcal{T}\}$$

be the set of periods of $\mathcal{T}$. We say that $\mathcal{T}$ admits an *algebraic coincidence* if there exist $N \in \mathbb{Z}_+$ and $\xi \in \mathcal{C}_i$ for some $i \leq \kappa$ for which $\xi + Q^N \Xi(\mathcal{T}) \subset \mathcal{C}_i$. It is known in [17] that $\mathcal{T}$ admits an algebraic coincidence if and only if $\mathcal{T}$ has a pure point spectrum.

With the assumption that $\mathcal{T}$ admits an algebraic coincidence, we define a topology on L and construct a completion H of the topological group L such that the image of L is a dense subgroup of H. This enables us to construct a cut-and-project scheme (CPS) such that each point set $\mathcal{C}_i$, $i \leq \kappa$, arises from the CPS. From the following lemma, we understand that the system $\{\alpha + \phi^n \Xi(\mathcal{T}) + \mathcal{K} : n \in \mathbb{Z}_+, \alpha \in L\}$ satisfies the topological properties for the group L to be a topological group [18–20].

Lemma 3 ([17] (Lemma 4.1))**.** *Let $\mathcal{T}$ be a primitive substitution tiling with an expansive map ϕ. Suppose that $\mathcal{T}$ admits an algebraic coincidence. Then, the system $\{\phi^n \Xi(\mathcal{T}) + \mathcal{K} : n \in \mathbb{Z}_+\}$ serves as a neighbourhood base for $0 \in L$ of the topology on L relative to which L is a topological group.*

For the topology on L with the neighbourhood base $\{\alpha + \phi^n \Xi(\mathcal{T}) + \mathcal{K} : n \in \mathbb{Z}_+, \alpha \in L\}$, we name *$\phi$-topology*. Let L_ϕ be the space L with ϕ-topology.

Let $L' = L/\mathcal{K}$. From [18] (Sections 3.4 and 3.5) and Lemma 3, there exists a complete Hausdorff topological group (H) of L' for which L' is isomorphic to a dense subgroup of the complete group H (see [6,21]). Moreover, there is a uniformly continuous mapping $\psi : L \to H$ which is the composition of the canonical injection of L' into H and the canonical homomorphism of L onto L'. Here $\psi(L)$ is dense in H and the mapping ψ from L onto $\psi(L)$ is an open map, where $\psi(L)$ is with the induced topology of the completion H. We can directly consider H as the Hausdorff completion of L vanishing $\mathcal{K}$.

4.2. P_ϵ-Topology

There is another topology on L which is equivalent to ϕ-topology under the assumption of algebraic coincidence.

Let $\{F_n\}_{n \in \mathbb{Z}_+}$ be a van Hove sequence and $\mathcal{T}', \mathcal{T}''$ be two tilings in $\mathbb{R}^d$, where $\mathbf{\Lambda}' = (\Lambda_i')_{i \leq \kappa}$ and $\mathbf{\Lambda}'' = (\Lambda_i'')_{i \leq \kappa}$ are representable Delone κ-sets of the tilings $\mathcal{T}', \mathcal{T}''$. We define

$$\rho(\mathcal{T}', \mathcal{T}'') := \lim_{n \to \infty} \sup \frac{\sum_{i=1}^{\kappa} \sharp((\Lambda_i' \triangle \Lambda_i'') \cap F_n)}{\mathrm{Vol}(F_n)}. \tag{11}$$

Here $\triangle$ is the symmetric difference operator. Let $P_\epsilon = \{t \in L : \rho(t + \mathcal{T}, \mathcal{T}) < \epsilon\}$ for each $\epsilon > 0$. For any $\epsilon > 0$, P_ϵ is relatively dense, if $\mathcal{T}$ admits an algebraic coincidence, then, [6,8,22,23]. In this case, the system $\{P_\epsilon : \epsilon > 0\}$ serves as a neighbourhood base for $0 \in L$ of the topology on L where L becomes a topological group. We call this topology P_ϵ-topology on L and indicate the space L with P_ϵ-topology by L_P (see [6,22] for P_ϵ-topology with the name of autocorrelation topology).

Proposition 1 ([17] (Propositions 4.6 and 4.7))**.** *Let $\mathcal{T}$ be a primitive substitution tiling. Assume an algebraic coincidence on $\mathcal{T}$, then the map $\iota : x \mapsto x$ from L_ϕ onto L_P is topologically isomorphic.*

Remark 1. From Proposition 1, L_P is topologically isomorphic to L_ϕ. Then, the completion of L_P is topologically isomorphic to the completion H of L_ϕ. We will identify the completion of L_P with H. Thus, $\varphi := \psi \cdot \iota^{-1} : L_P \to H$ is uniformly continuous, $\varphi(L_P)$ is dense in H, and the map φ from L_P onto $\varphi(L_P)$ is an open map where the latter is with the induced topology of the completion H. Hence, we can consider the CPS (5) with an internal space H that is a completion of L_P. We note that since $\mathcal{T}$ is repetitive, $\bigcap_{\epsilon>0} P_\epsilon = \mathcal{K}$ and $\mathcal{K} = \overline{\{0\}}$ in L_ϕ.

We observe that L_P and $\Psi(L)$ are all topologically isomorphic when the control point set $\mathcal{C}$ is a regular model κ-set in CPS (10).

Theorem 2. *Let $\mathcal{T}$ be a primitive Pisot family substitution tiling in $\mathbb{R}^d$ with an expansive map ϕ. Suppose that ϕ is diagonalizable, all the eigenvalues of ϕ are algebraic conjugates with multiplicity one, and there exists at least one algebraic conjugate λ of eigenvalues of ϕ for which $|\lambda| < 1$. If $\mathcal{C}$ is a regular model κ-set in CPS (10), then the internal space H which is the completion of L_ϕ with ϕ-topology is isomorphic to the internal space $\mathbb{K}$, which is constructed from using the conjugation map Ψ in (8).*

Proof. Since ϕ is an expansive map and satisfies the Pisot family condition, we first note that there is no algebraic conjugate γ of eigenvalues of ϕ with $|\gamma| = 1$.

We show that if $\Psi(t)$ is close to 0 in $\mathbb{K}$ for $t \in L$, then $\rho(t + \mathcal{T}, \mathcal{T})$ is close to 0 in H. Since every point set $\mathcal{C}_i$ is a regular model set from the assumption where $\mathcal{C} = (\mathcal{C}_i)_{i \le \kappa}$ and $\mathcal{C}_i = \Lambda(W_i)$ in the CPS (10), for $t \in L$

$$\begin{aligned} \rho(t + \mathcal{T}, \mathcal{T}) &= \lim_{n\to\infty} \sup \frac{\sum_{i=1}^{\kappa} \sharp(((t + \mathcal{C}_i) \bigtriangleup \mathcal{C}_i) \cap A_n)}{\mathrm{Vol}(A_n)} \\ &= \sum_{i=1}^{\kappa} \lim_{n\to\infty} \frac{\sharp(((t + \mathcal{C}_i) \bigtriangleup \mathcal{C}_i) \cap A_n)}{\mathrm{Vol}(A_n)} \\ &= \sum_{i=1}^{\kappa} (\theta(W_i \backslash (\Psi(t) + W_i)) + \theta(W_i \backslash (-\Psi(t) + W_i))), \end{aligned} \tag{12}$$

where θ is a Haar measure in $\mathbb{K}$ (see [24] [Theorem 1]).

We note that

$$\theta(W_i \backslash (s + W_i)) = \theta(W_i) - \mathbf{1}_{W_i} * \widetilde{\mathbf{1}_{W_i}}(s)$$

is uniformly continuous in $s \in \mathbb{K}$ [25] [Section 1]. So if $\Psi(t)$ converges to 0 in $\mathbb{K}$,then $\rho(t + \mathcal{T}, \mathcal{T})$ converges to 0 in $\mathbb{R}$.

On the other continuity, suppose that $\{t_n\}$ is a sequence such that $\rho(t_n + \mathcal{T}, \mathcal{T}) \to 0$ as $n \to \infty$. Then, for every $i \le \kappa$

$$\{\theta(W_i \backslash (\Psi(t_n) + W_i))\}_n \to 0, \quad n \to \infty.$$

Note that for large enough $n \in \mathbb{N}$, $W_i \cap (\Psi(t_n) + W_i) \ne \emptyset$, and so $\Psi(t_n) \in W_i - W_i$ for all $i \le \kappa$. From the fact that $W_i - W_i$ is compact, $\{\Psi(t_n)\}_n$ has a converging subsequence $\{\Psi(t_{n_k})\}_k$. For any such sequence, we define $t_0{}^* := \lim_{k\to\infty} \Psi(t_{n_k})$. Then

$$\theta(W_i \backslash (t_0{}^* + W_i)) = 0,$$

and thus $\theta(W_i{}^\circ \backslash (t_0{}^* + W_i)) = 0$ for each $i \le \kappa$. Hence $W_i{}^\circ \subset t_0{}^* + W_i$, and it implies $W_i \subset t_0{}^* + W_i$. On the other inclusion, $\lim_{k\to\infty} -\Psi(t_{n_k}) = -t_0{}^*$ and $\theta(W_i{}^\circ \backslash (-t_0{}^* + W_i)) = 0$. So, $W_i \subset -t_0{}^* + W_i$. Hence $W_i \subset t_0{}^* + W_i \subset t_0{}^* - t_0{}^* + W_i$ and $W_i = t_0{}^* + W_i$. This equality is for every $i \le \kappa$. Since $\mathbb{K}$ is isomorphic to $\mathbb{R}^{m-d}$, each model set W_i has irredundancy. Therefore $t_0{}^* = 0$. So all converging subsequences $\{\Psi(t_{n_k})\}_k$ converge to 0 and $\{\Psi(t_n)\}_n \to 0$ as $n \to \infty$.

This establishes the equivalence of the two topologies. By [18] (Proposition 5, Chapter 3, Section 3.3), there exists an isomorphism between H onto $\mathbb{K}$. □

The above theorem shows that the internal space H constructed from L_ϕ with ϕ-topology is isomorphic to Euclidean space $\mathbb{K}$ (i.e., $\mathbb{R}^{m-d}$).

It is known in [1,5,26] [Theorem 3.6] that unimodular irreducible Pisot substitution tilings in $\mathbb{R}$ with pure point spectrum give rise to regular model sets. We give a precise statement below.

Theorem 3 ([5] Remark 18.5)**.** *Let $\mathcal{T}$ be a primitive substitution tiling in $\mathbb{R}$ with expansion factor β being a unimodular irreducible Pisot number. Then $\mathcal{T}$ has a pure point spectrum if and only if for any $1 \leq i \leq \kappa$, each $\mathcal{C}_i$ is a regular model set in CPS (10).*

Corollary 1. *Let $\mathcal{T}$ be a primitive Pisot substitution tiling in $\mathbb{R}$ with an expansion factor β. Assume that there exists at least one algebraic conjugate λ of β for which $|\lambda| < 1$. If $\mathcal{T}$ has a pure point spectrum, then the internal space H which is the completion of L_ϕ with ϕ-topology can be realised by Euclidean space $\mathbb{R}^{m-1}$, where m is the degree of the characteristic polynomial of β.*

Proof. By Theorem 3, it is known that for a primitive Pisot substitution tiling in $\mathbb{R}$, if $\mathcal{T}$ has a pure point spectrum, then $\mathcal{C}$ is a regular model κ-set in CPS (10).

5. Conclusions

We constructed a natural cut-and-project scheme (10) where the control point sets are in the form of a module $\mathbb{Z}[\phi]\xi$. We showed that for 1-dimensional primitive Pisot substitution tilings, if they have pure point spectrum, then the abstract internal spaces given in [17] can actually be realised as Euclidean spaces. The questions still remains as to whether it holds for n-dimensional primitive substitution tilings. Substitution tilings are often used as mathematical models to understand the structures of physical materials and cut-and-project schemes with Euclidean spaces are used in simulation experiments in order to understand the pure point part of the spectrum. Here the results show that if one works on the simulation of 1-dimensional primitive Pisot substitution tilings with pure point spectrum, the corresponding representative point sets are always realised as model sets in a cut-and-project scheme with a Euclidean internal space.

6. Further Study

We are left with the following questions extending Theorem 2.

Question 1. *Can we replace the assumption of regular model κ-set by pure point spectrum? In other words, for a primitive Pisot family substitution tiling $\mathcal{T}$ in $\mathbb{R}^d$ with an expansion map ϕ, does the pure point spectrum of $\mathcal{T}$ imply that $\mathcal{C}$ is a regular model κ-set with a Euclidean internal space?*

Question 2. *Can the theorem be extended into the case where the multiplicity of eigenvalues of ϕ is not one?*

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